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ORIGINAL ARTICLE

Mathematical practices, in theory and practice

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Abstract
Descriptions of mathematical thinking have an extended lineage. Sometimes accurate and sometimes not, sometimes misinterpreted and sometimes not, characterizations of mathematical thought processes have inspired and at times misled people interested in designing or framing mathematics instruction. Challenges the field faces in conceptualizing mathematics instruction include: What can be warranted as legitimate mathematical practices? Which aspects of mathematical practices are relevant and appropriate for K-16 instruction? What kinds of support are necessary? What is viable at scale? This paper provides a description of relevant history from the Western literature, bringing readers up to the present. It then addresses two key issues related to contemporary curricula: the framing of the mathematical enterprise as being fundamentally inquiry-oriented and the need for curricula and instruction to reflect such mathematical values; and the characteristics of mathematical classrooms that support students' development as powerful mathematical thinkers. An emphasis is on problem solving, a major component of "thinking mathematically." The paper concludes with a description of practices that are currently under-emphasized in instruction and that would profit from greater attention.

Keywords Mathematical practices · Goals for instruction · Robust learning environments · Teaching for robust understanding framework

1 Introduction

This paper explores historical and contemporary issues related to descriptions of the practices of professional mathematicians and their relevance for mathematics teaching and learning. It begins with a selected review of work by major Western[1] mathematicians and philosophers whose tacit or explicit characterizations of mathematical practices have had a significant impact on mathematics education. At the end of each section through Sect. 6, I point to the relevance of each perspective and the issues it raises. A major focus is on problem solving, although the intended scope is broader – "thinking mathematically" includes problem posing, generalizing, and abstracting, for example. Sections 7 and 8 frame an approach aimed at helping students become powerful mathematical thinkers.

My point of departure is a conversation with a mathematician who studied under the "Moore Method" (Coppin, Mahavier, May & Parker 2009). In the Moore method's purest form, students are presented with mathematical definitions and asked to prove theorems. The students are barred from reading mathematics books or using other resources; the idea is for them to develop the mathematics themselves.

Moore's classes started as follows.

> 1. At the first meeting of the class Moore would define the basic terms and either challenge the class to discover the relations among them, or, depending on the subject, the level, and the students, explicitly state a theorem, or two, or three. Class dismissed. Next meeting: "Mr Smith, please prove Theorem 1. Oh, you can't? Very well, Mr Jones, you? No? Mr Robinson? No? Well, let's skip Theorem 1 and come back to it later. How about Theorem 2, Mr Smith?"[2] Someone almost always could do something. If not, class dismissed. (Halmos 1985, p. 255.)

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[1] Traditions outside the Western literature are beyond the scope of what can be covered here.

[2] I will note but not dwell on the maleness in Halmos's description and the descriptions of Moore's racism (see, e.g., https://www.math.buffalo.edu/mad/special/RLMoore-racist-math.html and https://en.wikipedia.org/wiki/Robert_Lee_Moore).

The next class began with the same questions. Students went to the board. If they did well, fine; if they erred, the class had to fix their mistakes. It soon became clear that – perhaps to avoid monologues by the strongest students – Moore called first on the students he perceived to be weakest. The result was an intensively intimidating atmosphere. The mathematician I spoke with thrived under that regime – but he also said that a number of very talented mathematics students withered under the pressure and decided to leave mathematics.

Herein lies the dilemma. The Moore method can be seen as a reductio ad absurdum of the proposition that specific mathematical practices such as proving should be at the core of mathematics instruction: students taught by Moore and his rigorous disciples got a 100% dose of such practices. Many of the students who swam in Moore's sink-or-swim environment went on to be high-powered mathematicians. The contributions to mathematics of Moore's 50 graduates and his 3840 descendants are clear and the Moore method has a devoted following. But many students sank in the high-pressure environment, and that is a great loss. One will never know what those people might have contributed to mathematics – and equally important, what damage was done to them personally.

This introductory story highlights the fact that values are central to considering which practices, to what degree, are appropriate for students. Key questions are: what are the "right" mathematical practices for students to learn, at what ages, in what ways; and what else needs to be part of the mix? In Sects. 7.2 and 8 I will address those questions directly.

Before continuing with a historical tour I note that mathematics education coalesced as a discipline in the 1960s (although mathematicians have long felt free to opine on educational issues). The first ICME was held in 1969, the journal *ZDM Mathematics Education* was first published (as *Zentralblatt für Didaktik der Mathematik*) in 1969 (Kaiser 2017), and Volume 1 of *Educational Studies in Mathematics* appeared in 1968. The *Journal for Research in Mathematics Education* was birthed in 1970, but it was a long and difficult birth (Johnson, Romberg & Scandura 1994). Prior to that, there were of course, mathematics educators: Fawcett (1938) is a classic example. But, writing on mathematics thinking, teaching, and curricula was largely the province of mathematicians and philosophers. Most work until the past half century focused on the nature of mathematical thought, with scant consideration given to the impact of mathematics learning on individuals.

2 The Meno dialog

The first recorded "Socratic lesson" in mathematics occurs in the Meno Dialogue (Plato, 2005). Against a larger philosophical backdrop – Socrates and Meno wrestle with definitions of virtue and the nature of knowledge and knowing – Socrates endeavors to demonstrate to Meno that humans, who have eternal and all-knowledgeable souls that transmigrate into their bodies at birth, can be induced to "recollect" their knowledge through careful questioning.

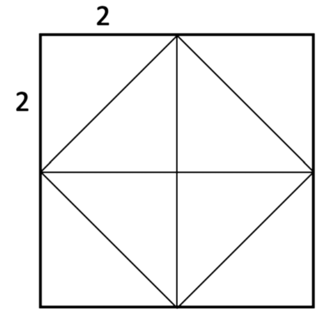

Fig. 1 The mathematical object discussed by Socrates and Meno's slave

Socrates asks Meno to select one of his slaves for the demonstration. The slave has minimal mathematical knowledge, limited to basic arithmetic and, for example, the understanding that a square that a square with sides of length 2 has area 2×2. Consider the 2×2 square in the upper left-hand corner of Fig. 1.

At Socrates' urging, the slave guesses that doubling the side of that square, which will produce Fig. 1, will double its area. With a series of extremely leading questions, Socrates gets the slave boy to acknowledge that the area of the "doubled" square is four times the area of the original square, not twice. He then argues, to Meno, that he has not *taught* the slave anything, but that the slave "will know it all without having been taught, only questioned, by finding knowledge within himself."

The Meno dialogue has served as the inspiration for "Socratic teaching" (and, see below, "inquiry based learning," "problem based learning," etc.). See https://www.socraticmethod.net/. It has been interpreted in a wide variety of ways – see, e.g., https://en.wikipedia.org/wiki/Socratic_method; https://www.criticalthinking.org/pages/socratic-teaching/606; https://www.thoughtco.com/what-is-the-socratic-method-2154875).

As I read the Meno, the dialog is a sham. The slave may or may not have learned something from the exchange, but there is no evidence, given Socrates' leading questions, that the slave "knew" the results beforehand. I could easily create a similar dialog with a student. I might ask the student to focus on the sums of 1, $1+3$, $1+3+5$, $1+3+5+7$, and $1+3+5+7+9$, asking if each one can be represented as the square of a whole number. Yes, they would say: the sums are 1, 4, 9, 16, 25, and that those sums are 1^2, 2^2, 3^2,4^2, 5^2. And what does that show? Not that the student "knew" that the sum of the first n odd numbers is n^2, but, rather, that I am a clever manipulator.[3]

This raises the question of questioning practices. Questions that lead students down a predetermined path are essentially closed, despite the appearance of being open. Yet, questions that lead students to conjecture, to pose problems, and to look for abstractions and generalizations can lead students into central mathematical practices. I expand on these thoughts in the concluding discussion.

3 Descartes

Descartes' contributions to philosophy, mathematics, and physics were monumental. Cartesian coordinates are named for Descartes' role in producing them. In fact, Cartesian coordinates were a small part of Descartes' (1954) grand plan, outlined in his draft *Rules for the direction of the mind.* The work was found posthumously; it was probably abandoned as Descartes found it impossible to complete.

Descartes' plan was for the *Rules* to have three books, each containing twelve rules. The thirty-six rules, altogether, would allow humans to solve all problems. The first book was completed, at least in draft. The outlines of nine rules were drafted for book 2; no trace of book 3 exists. To oversimplify, the first book was about what would, today, be called productive habits of mind – for example,

Rule V
The whole method consists entirely in the ordering and arranging of the objects on which we must concentrate our mind's eye if we are to discover some truth. We shall be following this method exactly if we first reduce complicated and obscure propositions step by step to simpler ones, and then, starting with the intuition of the simplest ones of all, try to ascend through the same steps to knowledge of all the rest.
(https://en.wikisource.org/wiki/Rules_for_the_Direction_of_the_Mind)

The grand plan was to produce methods to solve all simple mathematical problems in Book 1, reduce complex mathematical problems to families of simple problems in Book 2, enabling their via the methods in Book 1; and in Book 3 to find ways to represent all problems mathematically, at which point they could be solved by the methods in Books 2 and 1 respectively. Ambitious? Yes! Successful? As noted, the work was never completed. But, consider Cartesian coordinates as an example. They enable geometric objects to be represented in algebraic terms, and vice-versa – a powerful instantiation of Descartes' grand plan.

[3] See my formulation of this problem in Sect. 7.

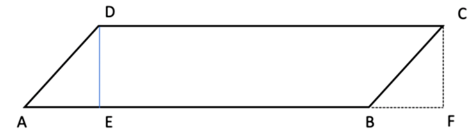

Fig. 2 The parallelogram in standard position

Of particular interest because they foreshadow Pólya and the theme of this issue are Descartes' ideas about powerful mathematical practices. Consider, for example, Rules XV–XVI, which would be labeled heuristic strategies in today's language:

Rule XV
It is generally helpful if we draw these figures and display them before our external senses. In this way it will be easier for us to keep our mind alert.

Rule XVI
As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.

Descartes' overall plan was expansive, concerned very much with being an effective mathematical thinker. As such, he was concerned with productive mathematical practices, and with what we would today call productive habits of mind (cf. Cuoco et al., 1996). Current work, of course, builds on these interesting philosophical ideas with empirical findings.

4 Wallas, Hadamard, Poincaré, Duncker and Wertheimer – the gestaltists

The mathematician in me resonates when I read Wertheimer (1945/59), Duncker (1945), Hadamard (1945), and Poincaré (1913). More than the authors of most contemporary curricula, they focused on important mathematical insights and ideas.

Perhaps the most famous example from the Gestaltists is Wertheimer's (1945/59) discussion of the "parallelogram problem." Wertheimer had visited a class in which students had learned to reproduce the proof that the area of a parallelogram is equal to its base times its height, and to compute examples thereof. The proof is suggested in Fig. 2, where the parallelogram is in standard position. Triangle AED is congruent to triangle BFC; "moving" triangle AED from its original position to the position occupied by triangle BFC results in the rectangle EFCD, so the parallelogram ABCD

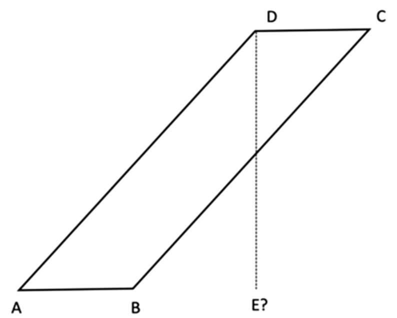

Fig. 3 A congruent parallelogram in non-standard position

and the rectangle EFCD have the same area (which, in the case of the rectangle is CD x DE, which are equal in extent to the base and height of the parallelogram).

Wertheimer asked for permission to quiz the children and, when it was granted, asked them to derive the area of the parallelogram in Fig. 3. It's essentially the same parallelogram, no longer in standard position.

From a mathematical (or what the Gestaltists would call a "structural") point of view, little of substance has changed: drawing the perpendicular from point D to side BC results in a figure akin to Fig. 2, and a similar argument can be made. However, the students were stymied: their focus, in procedural terms, was to drop the perpendicular from D to the base AB, and that procedure was no longer implementable.

The Gestaltists properly identified what was desired – perception of mathematical structure and a focus on big ideas. They also identified the main instructional problem: rote learning and a focus on procedures and replication do not result in structural, generative understanding. So far, so good. The challenge was their theoretical orientation.

As first posited by Wallas (1926) and then reified by the authors identified above, problem solving occurs in four phases:

1. Saturation – working on the problem until it becomes part of you and you can't do anything more with it consciously
2. Incubation – putting the problem away, and letting your subconscious work on it
3. Inspiration – in a flash, having the insight that suggests how things fit together
4. Verification – confirming the insight.

The Gestaltists offered anecdotal evidence in favor of this four-stage process. For example, Archimedes' discovery of buoyancy is the stuff of legend. The chemist F. A. Kekulé, struggling to discover the structure of benzene, dreamt of a snake biting its tail and was inspired to explore the possible ring structure of the compound. Hadamard (1945) writes of Poincaré having worked fruitlessly to understand the structure of Fuchsian functions, taking a day off to go on a geological excursion, and having an inspiration as he boarded the bus for the excursion.

Theoretically/methodologically speaking, these ideas were hard to verify and quite likely the result of "cherry picking" – the successes are heralded, but what about all the unremembered failures? Practically speaking, the theory was hard to convert into practice. As an example, consider Duncker's (1945) discussion of the "thirteen problem": Why are all six-digit numbers of the form *abc,abc* divisible by 13?

Duncker says that the difficulty disappears when the fundamental insight, that all such six-digit numbers are divisible by 1001, emerges from the subconscious. But in practical terms, what can one do in order to get such insights to emerge? Here the Gestaltists were of little help. They went no further than providing general suggestions – immersing oneself in the problem, taking a break, avoiding getting into a rut, etc.

As it happens, the thirteen problem is a personal favorite, which I discuss in Sect. 7.1. Now, in anticipation of Sect. 7.2 and what follows it, I want to turn to the human dimension of mathematics: who can do mathematics, and what image of mathematics is portrayed?

One could begin with Euclid, "There is no royal road to geometry." The meaning is usually taken to be "There are no shortcuts to learning geometry (or mathematics in general). One must apply oneself." But it also has a sarcastic edge – mathematics is (only) for those smart enough and dedicated enough to pursue it.

Poincaré, Gestaltist hero and mathematician/scientist of extraordinary prowess, makes this bias explicit in the *Foundations of science* (*1913*). Here he speaks of people in general:

> Now most men do not love to think, and this is perhaps fortunate when instinct guides them, for most often, when they pursue an aim which is immediate and ever the same, instinct guides them better than reason would guide a pure intelligence.… It is needful then to think for those who love not thinking, and, as they are numerous, it is needful that each of our thoughts be as often useful as possible. (Poincaré 1913, p. 365).

Here he speaks with regard to the challenges of pedagogy:

> How does it happen that so many refuse to understand mathematics – Is that not something of a paradox! Lo and behold – a science appealing only to the fundamental principles of logic … and there are people who find it obscure! and they are even in the majority! That they are incapable of inventing may pass, but that they do not understand the demonstrations shown them, that they remain blind when we show them a light which

seems to us flashing pure flame, this it is which is altogether prodigious. (Poincaré 1913, p. 430).

How one treats these issues (cf. the discussion of R.L. Moore) is essential. Leading questions as in the Meno do little or nothing for the slave as mathematical learner or thinker. Likewise, the essential mysticism of the Gestalt process provides us with no tools for inculcating mathematical habits of mind; indeed, it reifies the notion that some have mathematical talent and provides no entrée into mathematical thinking.

5 Pólya

To borrow Pólya's description of Descartes, Pólya was one of the very great. Pólya's contributions to mathematics were far greater than his contributions to mathematics education, and those alone were massive (cf. Alexanderson 2000, Alexanderson et al. 1987).

One of Pólya's advisees was Lakatos, parts of whose 1961 dissertation *Essays in the Logic of Mathematical Discovery* were later revised and produced as Lakatos's (1976) famous volume *Proofs and refutations.* A key idea of that volume, exemplified by the extensive discussion of the proof of the Euler characteristic, is that the first versions of definitions and proposed results are often not quite right, and need to be modified when counter-examples or other difficulties are found; hence mathematics is much more of an empirical/experimental discipline than a deductive one. Lakatos argues for an approach to instruction that is more heuristic than deductive.

In the 1920s most texts did (and most texts still do) set out the content for students to master, providing exercises along the way. As early as 1925 Pólya and his frequent mathematical collaborator Gábor Szego worked differently. Pólya & Szego's (1925) volume *Aufgaben und Lehrsätze aus der Analysis I (Problems and theorems in analysis I)* was a decidedly non-standard course in analysis: if you solved the problems and proved the theorems, you learned a significant amount of analysis. This can be seen as a robust antecedent of problem-based learning.

Pólya's more direct contributions to the field of problem solving began with *How to Solve it* (Pólya 1945) and continued with the two-volume sets *Mathematics and Plausible Reasoning* (Pólya 1954) and *Mathematical Discovery* (Pólya 1962/65). What is essential to understand is how much Pólya's ideas went beyond problem *solving* – even if one considers problem solving to be what Halmos (1980), another highly regarded mathematician who devoted significant efforts to mathematics education, called "the heart of mathematics" (Halmos 1980). For Pólya, mathematics was about inquiry; it was about sense making; it was about understanding how and why mathematical ideas fit together the ways they do. This explains why Pólya said that of the great historical mathematicians, the one who influenced him most was Euler: "Euler did something that no other great mathematician of his stature did. He explained how he found his results and I was deeply interested in that. It has to do with my interests in problem solving (Albers and Alexanderson, 1985, p. 251)". And, it was why the one movie of Pólya (1966) teaching is called *Let us teach guessing*. Inquiry, sense making, and exploring how things fit together are, in my opinion, key parts of what mathematical thinking (and practices) are all about. Pólya was heavily invested in helping people to learn to think mathematically; he taught summer institutes on "mathematical discovery" to teachers. All this contrasts significantly with the "philosophical" inquiries described earlier in this paper.

Inspired by Pólya's ideas (and following a decade of "back to basics" in the U.S. in the 1970s), the National Council of Teachers of Mathematics (1980, p. 1) recommended that "problem solving should be the focus of school mathematics in the 1980s." NCTM (1989) established the standards "movement" and re-emphasized problem solving as a fundamental goal of instruction. This was pursued in NCTM (2000) and, to some degree, in the "Common Core" (CCSSI 2000). However, as I see it, much was lost in the implementation of the ideas: "problem solving" became largely formulaic, and sense-making and mathematical thinking receded largely into the background. This was not simply an issue in the U.S. For broad overviews of how problem solving came to be interpreted and implemented around the world, see Törner, Schoenfeld, & Reiss (2008) and Santos-Trigo & Moreno-Armella (2013). Once again, the issue is how to open up the core mathematical ideas inherent in Pólya's work.

6 Influences on curriculum

There is a long history of groups specifying mathematics curricula, and of mathematicians weighing in on educational issues (cf. Poincaré, above). A major challenge is that most of what has been proposed is primarily about *content* rather than about *practices*. (In the US, this was the case in 1892 when the "Committee of Ten" recommended the standardization of the high school curriculum; it was the case seventy years later when the "new math" introduced ideas of modern mathematics into the curriculum.[4] Likewise, debates between formalists and intuitionists

[4] Interestingly, the science curricula introduced at the same (post-Sputnik) time, known as "hands on" curricula, were more process-oriented.

Fig. 4 A version of the "thirteen problem"

Take any three-digit number and write it down twice, to make a six-digit number. (For example, the three-digit number 789 gives us the six-digit number 789,789.) I'll bet you $1.00 that the six-digit number you've just written down can be divided by 7, without leaving a remainder.

OK, so I was lucky. Here's a chance to make your money back, and then some. Take the quotient that resulted from the division you just performed. I'll bet you $5.00 that quotient can be divided by 11, without leaving a remainder.

OK, OK, so I was very lucky. Now you can clean up. I'll bet you $25.00 that the quotient of the division by 11 can be divided by 13, without leaving a remainder.

Well, you can't win 'em all. But, you don't have to pay me if you can explain why this works.

(e.g., Hilbert and Poincaré) were mathematically important and interesting, but they did not have a direct impact on classroom practices. Among the exceptions, where there is attention to mathematical practices as well as content, one finds the work of the Freudenthal Institute, grounded in Realistic Mathematics Education (see, e.g., Freudenthal 1983; Gravemeier & Doorman 1999; Van den Heuvel-Panhuizen, 2003) and I.M. Gelfand's curriculum work (e.g., Gelfand, Glagoleva, & Schnol 1969; Gelfand & Saul 2001; Gelfand & Shen, 2003).

Two bodies of literature that attend to mathematical practices focus on mathematical modeling and on inquiry-based learning. Special notice must be given to Henry Pollak (1969, 1979, 2003), whose work inspired much of the international community's attention to modeling – a field that seems less integrated with mainstream mathematics education than it might be. There is a substantial literature on mathematical modeling, which has been the subject of an ICMI study (Blum, Galbraith, Henn, & Niss 2007), and which received comprehensive coverage in Volume 38, Issues 2 and 3, of ZDM (2006). On the surface, mathematical modeling demands (or at least offers affordances for) problem formulation and the use of representations (as well as reflecting on their adequacy). Likewise, inquiry-based learning (see Maass & Artigue 2013 for an overview and Maass et al. 2013 for an international summary) should provide affordances for meaningful mathematical inquiry and sense making. How much this happens, and how much of the approach is merely the formulaic use of problem solving techniques or the mechanical use of "the modeling cycle" is open to question.

7 Looking forward

Section 7.1 focuses on desirable mathematics classroom practices. Section 7.2 address the issue of supporting students' development as powerful mathematical thinkers.

7.1 Desirable mathematical practices

In this section I revisit and expand on two of the mathematics problems discussed above, using them as launching points for a discussion of where I think we need to head. Please note as you read the discussions that I think of problems as springboards for inquiry – that an attribute of a good problem is that it opens up avenues for conjecture, for making connections, for abstraction, generalization, and for new problems. Although the problems discussed below come from my problem solving courses, my intention is to discuss the kinds of tasks and discussions that I think should be central to all mathematics instruction.

I begin with the "thirteen problem," which we discuss the first day of my problem solving courses. Discussions of this and other problems continue over multiple days if more can be learned from them. The version I use is in Fig. 4.

This problem always provokes lively discussion because it offers opportunities for sense making, for representations, for making connections, and for taking ownership of the mathematics. The students may feel stymied at first but they ultimately generate a number of approaches, among them:

- Noting that dividing by 7, then 11, then 13, is the same as dividing by 7×11 x $13 = 1001$. How do you check your division? Multiply by 1001. Oh, that's why if works!
- Noting that the result of the final division is the 3-digit number you started with. How did that happen?
- Saying the six-digit number you'd created out loud and noticing the affordances for factoring, e.g., "seven hundred eighty-nine thousand, seven hundred eighty-nine" is seven hundred eighty-nine times one thousand plus seven hundred eighty-nine, which is $789 \times 1000 + 789 \times 1$ …
- Asking what the six-digit number really stands for. The students say that the number *abc, abc* is shorthand for $(a \times 10^5) + (b \times 10^4) + (c \times 10^3) + (a \times 10^2) + (b \times 10) + c$.

  I ask "what do you do when you see an expression that looks like this?" They respond "factor," which unlocks the problem.

Fig. 5 Pythagorean problems

> It's time to play with Pythagoras (well, with his ideas – not with him personally). For today, and perhaps a while after, we'll explore the Pythagorean theorem:
>
> *Given any right triangle with legs A and B, and hypotenuse C, then $A^2 + B^2 = C^2$.*
>
> Here I mean "explore" in a very broad sense. The idea is to take the Pythagorean theorem as a starting point, and ask questions. For example:
>
> Why is it true? Can we prove it in one, two, or three different ways? What if the triangle's not a right triangle? What can you say?
>
> We know that one solution in integers to the equation $A^2 + B^2 = C^2$ is the famous "3,4,5" right triangle. Are there other solutions in consecutive integers? In arithmetic sequence? In geometric sequence?
>
> Look at different solutions, in integers, to the equation $A^2 + B^2 = C^2$. Can you find anything interesting? Are there patterns in the relationships among A, B, and C? Are there infinitely many solutions to the equation in integers? Are there infinitely many "essentially different" solutions? Can you generate some of them? All of them?
>
> Are there Pythagorean quadruples: integers A,B,C,D such that $A^2 + B^2 + C^2 = D^2$?
>
> How about $A^2 + B^2 = C^2 + D^2$? Or . . . ??????

In class I highlight some of the strategies that have been used (e.g., working forwards in some solutions, working backwards in others; exploiting representations), the connections between approaches, and the issues of underlying mathematical structure. These discussions, in the spirit of Euler and Pólya, are aimed at understanding how and why things work.

One year a student who sat in the last row indicated that she had found a different solution. In an earlier discussion of the problem "what is the sum of the first 137 odd numbers?" I had suggested the strategy of systematically testing out some simple cases. For that problem, it meant asking, "what is the sum of the first 1 odd number, the first two odd numbers, the first 3 odd numbers, etc.?" This student said she he had done the same for the problem in Fig. 4, using the simple 3-digit numbers 001, 002, 003, etc. What she saw when she did was

$001{,}001 = 1001$

$002{,}002 = 2 \times 1001$

$003{,}003 = 3 \times 1001,$

which gave her enough insight to keep going. I had been using that problem for years and had never seen that approach; it was wonderful, and I told the class so.

It is worth noting that the student was a fourth year English major who told me that she had enrolled in the course because she "wanted to give math one last chance." She went on to gain confidence and do very well in the class.

In contrast to the Gestaltist mystification of the thirteen problem, the approach outlined here opens up the idea of systematic inquiry and of deliberate mathematical explorations; it points explicitly to ways by which students can uncover mathematical structure and connections. In addition, the student's experience described above points to the human dimensions of such inquiry, a theme I refer to in the Sect. 7.2.

Let us turn to the Pythagorean theorem. One of my favorite assignments is given in Fig. 5.

I want to start by highlighting the mathematical values, as well as the practices, suggested by the assignment. The fundamental stance taken is that of inquiry. Mathematics is about seeing how and why things fit together the way(s) they do. Students are given room to explore, to seek patterns, to conjecture – i.e., "Let us teach guessing" in the spirit of Pólya. This does not mean random guessing. Rather, one looks for ways to systematically expand one's knowledge; one evaluates conjectures and makes decisions accordingly. These are issues of strategic decision making and metacognition – see Schoenfeld 1985, 1987. Note also the relationship to problem posing (Brown & Walter 2005; Singer, Ellerton, & Cai 2015). This kind of problem framing makes it clear that mathematics is not about "getting the answer" – it's about developing and deepening one's understandings. What do you do when you've proved the Pythagorean theorem? You look for other proofs, for extensions, for applications. Different proofs may result in deeper or different understandings and to connections one has not made before. Building such understandings is part of the joy of mathematics. Students need to experience mathematical thinking as *generative*. See the discussion of the magic square problem in Sect. 8.

One year, when seeking all non-trivial integer solutions {A,B,C} to the Diophantine equation $A^2+B^2=C^2$ (that is, A, B, and C have no common factors), my students made an interesting conjecture. Soon after reading the problem they had listed all the Pythagorean triples they knew: {3,4,5}, (5,12,13}, {7,24,25}, {8,15,17}, and {12,35,37}. In the sample they generated, the hypotenuse differs from the larger leg by either 1 or 2, when the smaller leg is respectively odd or even. On the basis of this sample, they conjectured that

i. There are infinitely many Pythagorean triples of the form {x, y, y + 1};
ii. There are infinitely many Pythagorean triples of the form {x, y, y + 2};
iii. There are no others.

To my surprise, the students proved conjectures (i) and (ii). At that point a student asked, "If we prove there are no others, do we have a publishable theorem?"

Conjecture (iii) is false. There is a standard solution, which we discussed. But that's minor; we all make false conjectures. The point is that my students – in this class, student teachers who had no intention of being mathematicians –made two discoveries that were new to me and my colleagues. (In another course, the students did produce a result that got published.) The students were engaging in the practices of generating examples, perceiving relationships and making and testing conjectures according to the highest mathematical standards. *They were doing mathematics.*

This is a key point. The issue is not simply problem solving; It is about the use of problems as opportunities for developing mathematical practices and habits of mind, in any instructional context. It should be this way in all courses. A focus on using problems (broadly construed as opportunities for mathematical exploration, as illustrated in Fig. 5) raises the fundamental question,

> What do good problems (i.e., tasks and contexts that support the development of powerful mathematical practices, habits of mind, and rich discourse) look like?

In Schoenfeld (1991) I discussed my "problem aesthetic," a characterization of the properties of the kinds of problems I find instructionally valuable. Here is an updated distillation.

A. In general, I like problems that are easily understood and that do not require a lot of vocabulary or machinery as the "cost of admission." Undergraduates can start work on the four color problem and Fermat's last theorem without knowing too much background mathematics. Likewise, the explorations of the task in Fig. 5 can veer into deep mathematical territory. Much of the content in regular courses can be approached by "problematizing" the content in this way.
B. I tend to prefer problems that can be approached in a number of ways. Students tend to think that there is only one way to solve any given problem (usually the method the teacher has just demonstrated in class). In contrast, I want them to see mathematics as a realm for exploration. The possibility of multiple approaches also lays open issues of "executive" decisions – what directions or approaches should we pursue when solving problems, and why? Problems with multiple entry points allow more students to find handholds into the mathematics and are the seeds for rich student-to-student discourse, which is essential (cf. "group-worth problems," Cohen & Lotan 1997) and seeing the connections between multiple approaches deepens one's understanding.
C. The problems and their solutions should serve as introductions to important mathematical ideas. This can take place in at least two ways. Obviously, the topics and mathematical techniques involved in the problem solutions can be of agreed importance. And, problems can serve as fertile ground for learning problem solving techniques.
D. Problems should, if possible, serve as "seeds" for honest-to-goodness mathematical explorations. Open-ended problems provide one way to engage students in *doing* mathematics. Good problems lead to more problems – and if the domain is rich enough, students can start with the "seed" problem and proceed to make the domain their own (adapted from Schoenfeld 1991, p. 7).

These comments apply to all mathematical tasks. The idea is for tasks to support rich mathematical explorations and discourse – for them to serve as fertile ground for the development of mathematical practices. This perspective is entirely consistent with inquiry-based approaches to mathematics (see, e.g., Laursen & Rasmussen 2019; Maass, Artigue, Doorman, Krainer, & Ruthven 2013; Rasmussen, Wawro, & Zandieh 2015.

7.2 Supporting students' development as powerful mathematical thinkers

I now turn to the second main issue, the question of how we can support students' development as powerful mathematical thinkers. To frame the discussion I begin by characterizing two polar opposites. It may well be the case that the Moore method, which has its devotees and has undoubtedly produced many mathematicians of note, also destroyed the mathematical careers of many potentially fine mathematicians and drove many more people out of mathematics. A dose of pure mathematical practices was helpful for some and toxic for others. At the other end of the spectrum, the stories told in Sect. 7.1 discuss the mathematical experiences of people who had no intentions of pursuing careers producing mathematics. They show a graduating English major who was "giving math one last chance" finding her mathematical voice, and a class of pre-service teachers making truly interesting mathematical discoveries.

This raises the question: which mathematical practices, for whom? That, first and foremost, is a question of values. I believe that opportunities for real mathematical thinking as described in this paper can and should be made available to all. But, I must also point out that the question "which mathematical practices, for whom?" is also potentially misleading, because many people conceptualize the issue as a zero-sum game, along the lines of "Do I serve mathematics by honing the best talent, or do I serve the masses?" Simply put, both are possible: it is possible to offer rich mathematical instruction for all students (which, I note, enlarges the potential talent pool by enfranchising more potential mathematicians) *and* support those who wish to pursue mathematics. Framing instruction that does so has been the main focus of my research for the past decade.

The Teaching for Robust Understanding (TRU) project (2019) identifies the characteristics of classrooms from which all students emerge as knowledgeable and resourceful thinkers and problem solvers. Research indicates that five dimensions of classrooms practice are essential: the richness of the disciplinary content and practices (Dimension 1); Cognitive Demand, or the opportunity for students to engage in what has been called "productive struggle" (Dimension 2); equitable access to core content and practices for *all* students (Dimension 3); opportunities to develop a sense of agency, to make the mathematics their own, and to develop productive mathematical identities (Dimension 4); and, formative assessment, the degree to which student ideas are made public and responded to in productive ways (Dimension 5). It goes without saying that Dimension 1, the quality of the mathematics and the mathematical practices in which students engage, is crucial. But then the question is which students engage with the mathematics, and how they do so. Those are the other 4 dimensions of TRU, which has a central focus on the student's experience of the mathematics. There is by now ample evidence that each of the five dimensions of TRU is essential; that students who learn in classrooms that do well on the five dimensions of TRU tend to be knowledgeable and resourceful; and that because TRU embodies a set of principles but is not prescriptive, the framework can be adapted successfully for professional development in widely disparate contexts. (Schoenfeld 2013, 2019; Schoenfeld et al. 2020.)

Five dimensions of practice are, at first, a lot to keep track of. We have found that it is useful to conceptualize them as follows. At the forefront are the mathematics (Dimension 1) and *all* students' opportunities to engage with the core mathematics in ways that enable them to contribute and build on each other's ideas, thus building a sense of agency and positive disciplinary identities (Dimensions 3 and 4). One then asks, how can this be made to happen? A fundamental mechanism for doing so is making student thinking public, thus anchoring instruction in the students' current understanding and providing the grounds for productive struggle (Dimensions 5 and 2).

Evidence indicates that it is possible to configure classroom dynamics so that all students engage with core mathematical content and practices, in ways that support the development of productive mathematical identities – the English major and pre-service teachers discussed above can be taken as cases in point. A key point to understand is that opening up mathematics to more students does not necessarily make it easier (an argument made by those who feel that greater enfranchisement would disadvantage the "talented"); if the mathematics is opened up in the right ways it is made *richer,* because there are more connections to be made. As an example, compare the "lean" version of the "thirteen problem" in Sect. 4 with the more open version given in Sect. 7.1. The lean version may seem more challenging at first; but there is much richer mathematics, which includes the algebraic solution to the lean version, in a full discussion of the more open version. And, of course, there is nothing to prevent those students who get excited about this or other mathematics from digging more deeply. Making sure that every student can run an 8-min mile does not slow down those who can run faster.

I now believe that it is within the realm of possibility to do the kind of teaching discussed here – teaching that does well along the five dimensions of the TRU framework – at scale. The pedagogy in the Formative Assessment Lessons (Mathematics Assessment Project 2016), known as FALs, is entirely consistent with the TRU Framework (Burkhardt & Schoenfeld, 2019); the FALs have been shown to support teachers in building more TRU-like environments, with concomitant improvements in student performance (Herman et al. 2014; Research for Action 2015). The FALs span enough of the school year (50–60 days of instruction, at 5 grades) that it no longer takes a leap of imagination to envision their serving as the base for a body of instruction grounded in this kind of approach. However, there is a long way from possibility to reality, both within the classroom and regarding the contexts within which classroom activities take place. I can speak here with confidence regarding the U.S. Other national contexts differ, and they surely have their own issues as well.

An issue explored in Schoenfeld (2019) is that the mathematical practices encouraged by standards and related assessments are at far too granular a level: big ideas are lost when there is a focus on low-level detail. (See also Ma 2013). To give just one example, many students memorize the point-slope formula, two-point formula, slope-intercept formula, and two-intercept formula for lines in the plane, thinking they are all different in some fundamental way. The important understanding is that any two key pieces of information (among them, those identified above) determine the equation of a line, and that the different formulas are mere conveniences.

Finally, to avoid misconceptions: TRU is not prescriptive, in that it does not say *how* one should do well in the five TRU dimensions. Very different curricula, and instructors with very different styles, can excel in all five dimensions.

Reconceptualizing the curriculum to focus on big ideas is the easy part compared to what is needed in the other dimensions. Issues of formative assessment and cognitive demand (adjusting things so that students are engaged in productive struggle) are far more complex: they require a deep knowledge of the mathematics, so that the teacher can react in the moment to the ways classroom conversations are evolving, and keep the students productively engaged. They can be scaffolded by materials such as the FALs, however.

Moreover, the complexity of these issues pales in comparison to the complexity of issues related to equitable access and supporting student agency and identity. In classrooms in the U.S., this involves issues of race and power: the very moment students walk in the classroom door they are positioned by the way they look and act. Creating classroom environments that counter often unspoken assumptions, and that truly provide all students with opportunities to contribute meaningfully to the class's mathematical conversations – to be heard fairly and respectfully in generating ideas, reflecting on the group's work, and contributing to collective efforts – is a major challenge (Battey 2013; Battey et al., 2018; Martin 2006, 2009; McGee & Martin 2011; Shah 2017). Mathematics classroom practices and norms (Cobb & Yackel 1996; Yackel & Cobb 1996) shape each student's experience of the mathematics and what they take from it, personally as well as mathematically. And that, in the U.S., is just the beginning: in some schools, you can tell which mathematical track (stream) a class is in by the racial composition of the students in it; and there are huge disparities in the resources available to schools and school districts across the U.S., as a function of demographics. Such issues go beyond those of classroom practices, but they need to be mentioned.

8 Discussion

Reflecting on nearly 50 years of Research and development in this arena, I conclude with a list of pedagogical/mathematical practices that I think are under-emphasized and need to move to center stage.

- Establishing a climate of inquiry, in which mathematics is experienced as a discipline of exploration and sense making.

  One of the problems I use in the first day of my problem solving courses is to have the students work a 3×3 magic square. When they have done so, I ask if we are done. "Yes," they chorus. I say, "No we're not. What about placing, say, the integers 2 through 10, or 21 through 30?" When they have dealt with those problems, I ask if we are done. "Yes," they chorus once more. I say, "No we're not. What about the integers 20, 40, 60, …, 180? Or the integers 21, 41, 61, …, 181? The students generalize to the fact that if you think of any magic square as a matrix, M, then aM, and M + b; and aM + b are also magic squares. That ties up this series of mini-explorations neatly. I ask, "OK, are we done?" Yes, they say. I say, "No we're not. In the original magic square the sum of each row, column, and diagonal was 15. Suppose I give you a random integer. Is there a magic square whose rows, columns, and diagonals sum up to that integer? They prove that such integers must be divisible by 3, in which case such magic squares exist.

  Once again I ask, are we done? A student throws up his hands in mock exasperation and says, "We're NEVER done!" That, I say, is precisely the point. The way mathematics gets built is by consistently seeking to extend what we know through generalizations, abstractions, and pursuing issues of structure. Euclid's fifth postulate is a perfect example of structural concerns. From the time of Euclid, five postulates sufficed to define Euclidean geometry, for all practical purposes. But, for 2000 years, mathematicians tried to see if the dependence on five postulates could be reduced to a dependence on four. Eventually, non-Euclidean geometries were found. Mathematics is about problematizing. It's true: we're never done.

  A few weeks later the same student asked me why I was no longer asking "Are we done?" I told him that I didn't have to – the class was now running with mathematical ideas in the ways I'd hoped.
- Focusing on big ideas, and not losing the forest for the trees.

  I will not dwell on this, having devoted significant space to the ideas in Schoenfeld (2019). A key idea is *generativity*: what students learn and remember, should be sufficient to allow them to regenerate what they need, long after instruction is over. An indelible memory (literally more than 50 years ago) is my undergraduate probability teacher starting to write the statement of the binomial theorem, stopping, and saying "I always forget the statement, but it doesn't matter – it's so easy to derive the result." She reasoned her way through the theorem, and then wrote the statement at the top.

I should note that I consider heuristic strategies to be big ideas. The key ideas for approaching problem solving as curricular content were established, as existence proof, as long ago as Schoenfeld (1985).

- Making student thinking central to classroom discourse.

  Formative assessment depends on knowing what students think! Having student thinking be public is the best way to understand current student understandings, and for students and teacher alike to refine them (see Burkhardt & Schoenfeld, 2019). Moreover, the key to the development of a student's mathematical agency and identity, their mathematical authority (in the sense of authorship) and their ownership of what is produced is that student's contributions to core ideas and refining them. I hasten to add that there are multiple ways to contribute to collective work, among them: generating ideas, testing ideas and reflecting on them, seeking connections, clarifying and organizing contributions, and writing things down clearly.
- Insuring that classroom discourse is respectful and inviting.

  Students will not contribute to classroom discourse unless they feel it is safe to do so. As noted above, issues of race and power are likely to be in play in a classroom before any student says a single word; positionings in discourse are subtle but powerful (see, e.g., Langer-Osuna 2011, Reinholz & Shah 2018, Wood 2013). This means that the ways students are positioned – not just by the teacher but by other students as well – are critical. Finding ways to invite students into classroom discourse, whether via critically relevant pedagogy (e.g., Brown-Jeffy & Cooper 2011, Gay 2018, Ladson-Billings 1994, 1995, Palmer 1998), via relational teaching (e.g., Franke, Kazemi, & Battey 2007), or via any other mechanism one can use, is a beginning; making sure that students' contributions (even if not seemingly correct) are seen as enriching classroom discussions is essential. Cycling back to the first bullet, note that in a climate of inquiry, the vast majority of suggestions for making progress on challenging problems are sure to be flawed – as is the case with the vast majority of professional mathematicians' attempts to work at the cutting edge. It's only after many flawed attempts that the final product is polished.

In these final comments I have tried to focus on big ideas. The first two bullets pertain largely to Dimension 1 of the TRU framework, in an attempt to make sure that classroom norms and practices focus on what counts mathematically. The third and fourth bullets are at the heart of Dimensions 3 and 4. Our classrooms must be environments in which all students are empowered to engage meaningfully in mathematical practices, for such engagement is the source of agency and identity. Those two bullets also provide much of the substance behind dimensions 5 and 2: formative assessment is enabled when student thinking is made public, and work in students' zone of proximal development is the locus of productive struggle. Attention to such issues in curriculum and pedagogy will provide students with deeper opportunities to engage meaningfully with mathematics, and become more agentive and deeper mathematical thinkers.

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References

Alexanderson, G. (Ed.) (1987). Pólya commemorative issue. *Mathematics Magazine,* 60(5).

Albers, D., & Alexanderson, G. (1985). *Mathematical People*. Boston: Birkhäuser.

Alexanderson, G. (Ed.). (2000). *The random walks of George Pólya*. Washington DC: Mathematical Association of America.

Alexanderson, G. L., Chung, K. L., Boas, R. P., Lange, L., Lehmer, D. L., Schattschneider, D., et al. (1987). George Pólya—obituary. *Bulletin of the London Mathematics Society, 19*, 559–608.

Battey, D. (2013). "Good" mathematics teaching for students of color and those in poverty: the importance of relational interactions within instruction. *Educational Studies in Mathmathematics, 82*, 125–144. https://doi.org/10.1007/s10649-012-9412-z.

Battey, D., Leyva, L., Williams, I., Belizario, V., Greco, R., & Shah, R. (2018). Racial (mis)match in middle school mathematics classrooms: relational interactions as a racialized mechanism. *Harvard Educational Review, 88*(4), 455–483.

Blum, W., Galbraith, P., Henn, H.-W., & Niss, M. (Eds.) (2007). Applications and modelling in mathematics education**.** The 14th ICMI study. New York: Springer. ISBN: 978-0-387-29820-7.

Brown, S., & Walter, M. (2005). *The art of problem posing*. Mahwah: Erlbaum.

Brown-Jeffy, S., & Cooper, J. E. (2011). Toward a conceptual framework of culturally relevant pedagogy: An overview of the conceptual and theoretical literature. *Teacher Education Quarterly, 38*(1), 65–84.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3–4), 175–190.

Cohen, E. G., & Lotan, R. A. (Eds.). (1997). *Working for equity in heterogeneous classrooms: Sociological theory in practice*. New York: Teachers College Press.

Common Core State Standards Initiative (CCSSI). (2010). Common core state standards for mathematics. https://www.corestandards.org/Math/Practice.

Coppin, C. A., Mahavier, W. T., May, E. L., & Parker, E. (2009). *The Moore method*. Washington, DC: Mathematical Association of America. ISBN-13: 978-0883851852; ISBN-10: 0883851857.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior, 15*, 375–402.

Descartes, R. (1954) *Rules for the direction of the mind.* A translation by Elizabeth Anscombe and Peter Thomas Geach is available at

https://en.wikisource.org/wiki/Rules_for_the_Direction_of_the_Mind; also at https://www.gutenberg.org/files/59/59-h/59-h.htm.
Duncker, K. (1945). On problem-solving (L. S. Lees, Trans.). *Psychological Monographs, 58*(5), i-113. https://dx.doi.org/10.1037/h0093599.
Fawcett, H. (1938). The nature of proof. (Thirteenth Yearbook of the National Council of Teachers of Mathematics.) New York: Teachers College.
Franke, M., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. Lester (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (2nd ed., pp. 225–256). Charlotte: Information Age Publishing.
Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Reidel Publishing.
Gay, G. (2018). *Culturally responsive teaching: Theory, research, and practice* (3rd ed.). New York: Teachers College Press.
Gelfand, I. M., Glagoleva, E., & Schnol, E. (1969). *Functions and graphs*. Mineola: Dover.
Gelfand, I. M., & Saul, A. (2001). *Trigonometry*. Boston: Birkhauser.
Gelfand, I. M., & Shen, A. (2003). *Algebra*. Boston: Birkhauser.
Gravemeier, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*, 111–129.
Hadamard, J. (1945). *The psychology of invention in the mathematical field*. Princeton: Princeton University Press.
Halmos, P. (1980). The heart of mathematics. *American Mathematical Monthly, 87*, 519–524.
Halmos, P. (1985). *I want to be a mathematician*. New York: Springer.
Herman, J., Epstein, S., Leon, S., La Torre Matrundola, D., Reber, S., & Choi, K. (2014). *Implementation and effects of LDC and MDC in Kentucky districts* (*CRESST Policy Brief No. 13*). Los Angeles: University of California, National Center for Research on Evaluation, Standards, and Student Testing (CRESST).
Johnson, D. C., Romberg, T. A., & Scandura, J. M. (1994). The Origins of the JRME: A retrospective account. *Journal for Research in Mathematics Education, 25*(6), 560–582.
Kaiser, G. (2017). On the occasion of the decennial publication of ZDM Mathematics Education by Springer. *ZDM Mathematics Education, 49*(1), 1–3.
Ladson-Billings, G. (1994). *The dreamkeepers: Successful teachers of African American children*. San Francisco: Jossey-Bass.
Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Research Journal, 32*(3), 465–491.
Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.
Langer-Osuna, J. (2011). How Brianna became Bossy and Kofi came out smart: Understanding the trajectories of identity and engagement for two group leaders in a project-based mathematics classroom. *Canadian Journal of Science, Mathematics and Technology Education*, 11(3), 207–225.
Laursen, S., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics education. *International Journal of Research in Undergraduate Mathematics Education, 5*(1), 129–149.
Ma, L. (2013). A critique of the structure of US elementary school mathematics. *Notices of the AMs, 60*(10), 1282–1296.
Maass, K., & Artigue, M. (2013). Implementation of inquiry-based learning in day-to-day teaching: a synthesis. *ZDM—The International Journal on Mathematics Education, 45*, 779–795.
Maass, K., Artigue, M., Doorman, L.M., Krainer, K., & Ruthven, K. (Eds.) (2013). Implementation of Inquiry-Based Learning in Day-to-Day Teaching. *ZDM—The International Journal on Mathematics Education*, 45(6). ISSN: 1863-9690 (print) 1863-9704 (online).
Martin, D. B. (2006). Mathematics learning and participation as racialized forms of experience: African American parents speak on the struggle for mathematics literacy. *Mathematical Thinking and Learning, 8*(3), 197–229.
Martin, D. B. (2009). Researching race in mathematics education. *Teachers College Record, 111*(2), 295–338.
Mathematics Assessment Project. (2016). *Formative Assessment Lessons*. https://www.map.mathshell.org/lessons.php.
McGee, E., & Martin, D. B. (2011). "You would not believe what I have to go through to prove my intellectual value!" Stereotype management among academically successful Black mathematics and engineering students. *American Educational Research Journal, 48*(6), 1347–1389.
National Council of Teachers of Mathematics. (1980). *An agenda for action*. Reston: NCTM.
National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston: NCTM.
National Council of Teachers of Mathematics. (2000). *Principles standards for school mathematics*. Reston: NCTM.
Plato. (2005). *Meno and other dialogues* (R. Waterfield, Trans.). Oxford: Oxford University Press.
Palmer, P. (1998). *The courage to teach*. San Francisco: Wiley.
Pollak, H. (1969). How can we teach applications of mathematics? *Educational Studies in Mathematics, 2*(2–3), 393–404.
Pollak, H. (1979). The interaction between mathematics and other school subjects. In UNESCO (Eds.), *New Trends in Mathematics Teaching IV,* pp. 232–248. Paris: UNESCO.
Pollak, H. (2003). A history of the teaching of modelling. In G. Stanic & J. Kilpatrick (Eds.), *A History of School Mathematics* (pp. 647–671). Reston: NCTM.
Pólya, G. (1945). *How to solve it*. Princeton University Press: Princeton.
Pólya, G. (1954). *Mathematics and plausible reasoning* (Volume 1, Induction and analogy in mathematics; Volume 2, Patterns of plausible inference). Princeton: Princeton University Press.
Pólya, G. (1962, 1965/1981). *Mathematical Discovery* (Volume 1, 1962; Volume 2, 1965). Princeton: Princeton University Press. Combined paperback edition, 1981. New York: Wiley.
Pólya, G. (1966). *Let us teach guessing*. Washington DC: Mathematical Association of America. Also available at https://vimeo.com/48768091.
Pólya, G., & Szego, G. (1925). *Aufgaben und Lehrsätze aus der Analysis I*. Berlin, Germany: Springer. An English version, *Problems and theorems in analysis I*. (D. Aeppli, trans.), was published by Springer (New York) in 1972.
Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Educational Studies in Mathematics, 88*(2), 259–281. https://doi.org/10.1007/s10649-014-9583-x.
Reinholz, D. L., & Shah, N. (2018). Equity analytics: A methodological approach for quantifying participation patterns in mathematics classroom discourse. *Journal for Research in Mathematics Education, 49*(2), 140.
Research for Action. (2015). *MDC's Influence on Teaching and Learning*. Philadelphia, PA: Author. Retrieved from https://www.researchforaction.org/publications/mdcs-influence-on-teaching-and-learning/.
Santos-Trigo, L., & Moreno-Armella, L. (Eds.). (2013). International perspectives on problem solving research in mathematics education, a special issue of *The Mathematics Enthusiast,* 10(1&2).
Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic Press.
Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189–215). Hillsdale: Erlbaum.
Schoenfeld, A. H. (1991). What's all the fuss about problem solving? *Zentrallblatt fur Didaktik der Mathematik, 91*(1), 4–8.
Schoenfeld, A. H. (1994). What do we know about mathematics curricula? *Journal of Mathematical Behavior, 13*(1), 55–80.

Schoenfeld, A. H. (2013). Classroom observations in theory and practice. *ZDM—The International Journal on Mathematics Education, 45*, 607–621. https ://doi.org/10.1007/s1185 8-012-0483-1.

Schoenfeld, A. H. (2019). Reframing teacher knowledge: A research and development agenda. *ZDM: Mathematics Education.* https :// doi.org/10.1007/s11858-019-01057-5.

Schoenfeld, A. H., Baldinger, E., Disston, J., Donovan, S., Dosalmas, A., Driskill, M., Fink, H., Foster, D., Haumersen, R., Lewis, C., Louie, N., Mertens, A., Murray, E., Narasimhan, L., Ortega, C., Reed, M., Ruiz, S., Sayavedra, A., Sola, T., Tran, K., Weltman, A., Wilson, D., & Zarkh, A. (2020). Learning with and from TRU: Teacher educators and the teaching for robust understanding framework. In K. Beswick (Ed.), International Handbook of Mathematics Teacher Education, Volume 4, *The Mathematics Teacher Educator as a Developing Professional.* Rotterdam: Sense Publishers (**in press**).

Shah, N. (2017). Race, ideology, and academic ability: A relational analysis of racial narratives in mathematics. *Teachers College Record, 119*(7), 1–42.

Singer, F. M., Ellerton, N. F., & Cai, J. (Eds.). (2015). *Mathematical Problem Posing: From Research to Effective Practice*. New York: Springer.

Törner, G., Schoenfeld, A. H., & Reiss, K. (Eds.). (2008). Problem solving around the world—Summing up the state of the art. Special issue of the *Zentralblatt für Didaktik der Mathematik*, *39*(5–6).

Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*, 9–35.

Wallas, G. (1926). *The art of thought*. Kent: Solis Press.

Wertheimer, M. (1945). *Productive thinking*. New York: Harper & Row.

Wood, M. B. (2013). Mathematical micro-identities: Moment-to-moment positioning and learning in a fourth-grade classroom. *Journal for Research in Mathematics Education, 44*(5), 775–808. https://doi.org/10.5951/jresematheduc.44.5.0775.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458–477.

ZDM. (2006). Volume 38, Issue 2 (Part A) & Issue 3 (Part B): Special issue on *Mathematical Modeling Research around the World*.