Why Are Learning and Teaching Mathematics So Difficult?

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Abstract

Decades ago Hans Freudenthal referred to the school mathematics experienced by most students as the "fossilized remains" of reasoning processes. Indeed, the facts and procedures of school mathematics may seem as frightening to some as the fossilized remains of a tyrannosaurus rex, although they are empty; like dinosaur skeletons, they bear only partial resemblance to the real thing. The challenge is to see the substance behind the structure and to understand how the mathematics fits together. That is a matter of mathematical thinking, reasoning, and problem-solving – the *how* and the *why* beneath the fossilized surface. Opportunities for such understandings are accessible through mathematical sense making, but they are rare in schools. This chapter indicates that there is more to learning and

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M. Danesi (ed.), *Handbook of Cognitive Mathematics*,
https://doi.org/10.1007/978-3-030-44982-7_10-1

understanding mathematical content and practices than it would appear. Moreover, understanding mathematics is only one component of effective or "ambitious" teaching – better framed as the creation of mathematically rich and equitable learning environments. The challenge is to create robust learning environments that support every student in developing not only the knowledge and practices that underlie effective mathematical thinking, but that help them develop the sense of agency to engage in sense making. This implicates issues of race and equity, which are a challenge not only in classrooms but in society at large; structural and social inequities permeate the schools. Major obstacles to addressing the challenges of powerful mathematics within schools include a general absence of curricular support for rich and meaningful mathematics, instructional practices that do not invite students into mathematics, assessments that fail to focus on thinking, professional development that focuses on what the teacher does rather than the students' learning opportunities and experiences, and a vastly inequitable cultural context both outside and inside schools. This chapter points to existence proofs that at least some these challenges can be addressed, while documenting the substantial challenges to making progress at scale.

Keywords

Ambitious instruction · Bias · Formative assessment · Problem-solving · Summative assessment · Teaching for robust understanding · Sense making · Systemic inequities · Thinking mathematically

Introduction

Mathematics has a singular place in Western schooling and folklore. It is seen as frightening: math anxiety is a highly discussed phenomenon, returning more than "about 93,800,000" Google hits in less than half a second. It became an issue of pop culture when "Teen Talk Barbie," a mechanical Barbie doll with a voice box (https://en.wikipedia.org/wiki/Teen_Talk_Barbie), gained notoriety because one of her pre-programmed phrases was "math class is tough." People talk about there being a "math gene," in the sense that "either you have it or you don't"; the blurb for Keith Devlin's (2000) book *The Math Gene* says "Why is math so hard? . . . If there's some inborn capacity for mathematical thinking—which there must be, otherwise no one could do it – why can't we all do it well?" In the USA, once mathematics became an optional rather than required subject, "data from enrollment in mathematics courses reveal that roughly half the students leave the pipeline every year" (National Research Council, 1989). Moreover, attrition rates for Brown and Black students were consistently higher than for Whites, and, whatever the technical validity of the tests that generate them, racial and ethnic achievement gaps in mathematics continue to be a problem and an obstacle to the equitable enfranchisement of minoritized students (Lee, 2002; Moses, 2001). In turn, these differences feed into racial and other stereotypes that exacerbate the underlying conditions that cause them (Martin,

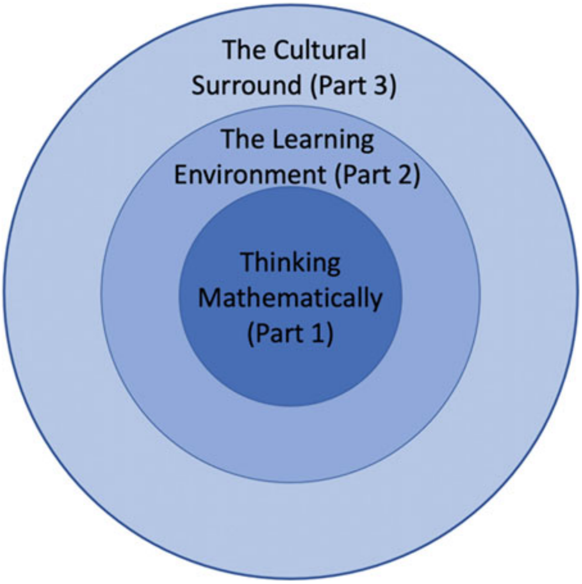

Fig. 1 The structure of this chapter, reflecting what matters in mathematics learning and teaching

2009; Sengupta-Irving & Vossoughi, 2019). In short, we have a problem with mathematics. The challenge is to explain, why are learning and teaching mathematics so difficult?

This chapter characterizes the challenges of understanding and teaching mathematics by way of three expanding circles, which are illustrated in Fig. 1.

Part 1 concerns the nature of mathematical thinking. Coming to understand mathematics means not only developing a grasp of facts, procedures, and concepts, but learning to think in and with the discipline, internalizing disciplinary habits of mind and the practices of doing mathematics (Schoenfeld, 2017). As discussed in the first main section of this chapter, robust mathematical thinking involves: understanding and enacting both the content and practices of the discipline; being able to employ a wide range of problem-solving strategies; employing monitoring and self-regulation and other metacognitive processes effectively; and having belief systems that support the use of one's mathematical knowledge and understandings (Schoenfeld, 1985, 1992). There is evidence that all of K-12 mathematics (and at least some mathematics at the college level) can be taught as a sensemaking enterprise; moreover, when students experience it as such, they retain/regenerate their understandings longer and more deeply (Bell, 1993; Swan, 2006).

The importance of content and processes, problem-solving, and metacognition and belief systems has been recognized for at least 35 years (Schoenfeld, 1985, 1992), and a "standards movement" emphasizing mathematical processes began in 1989. Yet, little of that sense making has made its way into classrooms. For decades, a focus on skills and on accountability at a fine-grained level has obscured the big ideas and made it challenging to see mathematics as coherent and sensible. Indeed, with the advent of high stakes testing, moves in that direction may have been stultified. The first main part of the chapter elaborates on these issues.

The second main part of this chapter problematizes the first, expanding the set of issues discussed to consider what occurs in the learning environment and what impact that has on students. In brief, the idea is that the impact of instruction extends far beyond students' coming to grips with the content and practices of a discipline. To give just one example, consider the ways in which students are positioned by each other and the teacher (Herbel-Eisenmann et al., 2015; McDermott, 1996). If, for whatever reason (e.g., race or ethnicity, gender, perceived abilities), a student is positioned as being mathematically less capable than others, that student may be deprived of opportunities for meaningful engagement with content and practices, directly affecting such knowledge; but also, such comments may shape the student's beliefs about their mathematical ability, which in turn affect agency and mathematical identity. In these ways, the overall classroom environment shapes individuals' mathematical outcomes, including belief systems (cf. Part 1), thus expanding the set of concerns regarding mathematical outcomes and difficulties.

Research indicates that classrooms from which students emerge as knowledgeable and resourceful thinkers and problem solvers do well along the following five dimensions: (1) the richness of the mathematical content and practices in which students have the opportunity to engage; (2) the degree to which students have opportunities to engage in "productive struggle" – sense making in what is known as the zone of proximal development (Vygotsky, 1986); (3) opportunities for *every* student to engage meaningfully with the content and practices of the discipline; (4) ways in which the classroom fosters student engagement that supports them in developing a sense of agency, making the mathematics their own, and developing positive disciplinary identities (Engle 2011); and (5) formative assessment (Black & Wiliam, 1998, Burkhardt & Schoenfeld, 2019) – student thinking being aired publicly, and the classroom environment responding adaptively to "meet the students where they are" (Schoenfeld, 2014, 2020). It should be apparent that addressing all of these issues – orchestrating "robust" or "ambitious" learning environments – is an extremely demanding task, even under supportive conditions such as having well designed curricula, aligned assessment, effective professional development and consistent policy messages, within a supportive political context.

All of the preceding, however, refers to what takes place within classroom walls. Part 3 of this chapter deals with the obvious fact that classrooms are microcosms of the larger society and that aspects of that larger society make teachers' work in trying to achieve ambitious or robust instruction that much more difficult. This happens in at least two fundamental ways. The first is resource allocation. Whether it was a matter of the spuriously labeled "separate but equal" facilities for Blacks and Whites by law in the USA prior to the 1954 Brown v. Board of Education decision by the US Supreme Court (Orfield & Eaton, 1996) or structural racism (Wilkerson, 2020), or ongoing economic discrimination, minoritized groups have long suffered from the unequal distribution of resources – e.g., dilapidated school buildings, limited curricular options and outdated texts, large student-to-teacher ratios, and high teacher turnover and low numbers of credentialed teachers (Kozol, 1992), the deck has been consistently stacked against marginalized subpopulations of society. Second, biases do not stop at the classroom door; classroom dynamics often replicate societal biases in subtle but powerful ways. Teachers who unwittingly call on boys more than girls or ask boys higher-level questions than girls reflect societal biases (AAUW 1992), as do teachers who hold high expectations of "model minorities" (Nasir & Shah, 2011) or low expectations of other ethnic groups (Shah 2017). The impact of these expectations on performance is well known (cf. stereotype threat, Steele & Aronson, 1995). More fundamentally, continuous exposure to such biases – whether from teachers or fellow students – shapes beliefs, and thus has a fundamental impact on teaching and learning.

In sum, the expanding set of issues described respectively in Parts 1, 2, and 3 of this chapter – the broad challenges of developing facility with mathematical content and practices; the complex challenges of creating robust teaching environments; and the additional difficulties raised by the fact that schools reflect the structural and other inequities of society at large – contribute in major ways to the fundamental challenges of teaching and learning of mathematics.

Part 1. The Nature of Mathematical Thinking

The main focus of this section, which emphasizes developments in the field's understandings over the past half century, is on what it means to think mathematically. Philosophical debates about the nature of mathematics are beyond the scope of this chapter. Nor does this chapter engage questions of whether there is a "math gene," beyond noting that that the assumption that mathematical abilities are fixed is predominantly Western and not shared in parts of Asia, or whether people are hardwired to do "mathematics" ("mathematics" in quotes because what is typically considered to be mathematics in such arguments is an impoverished subset of mathematical thinking). The key question is, what characterizes productive and powerful mathematical engagement?

A revolution in the field's understanding of mathematical thinking and learning – then called the "cognitive revolution," later expanding to the "learning sciences approach" – began in the mid-1970s, with the emergence of cognitive science as a discipline. It coincided with increased attention in the mathematics education community on issues of mathematical thinking and problem-solving. The history, condensed to a few paragraphs, is as follows.

There was a "democratization" of access to mathematics over the course of the twentieth century. In 1890, fewer than 10% of the 14-year-olds in the USA attended high school, and only 3.5% of the 17-year-olds graduated. In the early 1940s, almost three-fourths of the children of age 14–17 attended high school, and nearly half of the 17-year-olds graduated (Stanic, 1987, p. 150). In the 2017–2018 school year, the national adjusted cohort graduation rate for public high school students was 85% (https://nces.ed.gov/programs/coe/indicator_coi.asp). Mathematics, once the study of the elite, became required content for most students through at least 10th grade. Socially, mathematics became a "filter" at all levels. High failure rates in high school mathematics – and substantially higher failure rates for minoritized students – were major factors in students dropping out of school. Beyond that, students who graduated from high school with minimal mathematics coursework lacked the prerequisites for scientific majors when they did go to college. For these reasons among others, Moses (2001) framed access to mathematics as a civil rights issue:

> Today . . . the most urgent social issue affecting poor people and people of color is economic access. In today's world, economic access and full citizenship depend crucially on math and science literacy. I believe that the absence of math literacy in urban and rural communities throughout this country is an issue as urgent as the lack of Black voters in Mississippi was in 1961. (Moses, 2001, p. 5)

Mathematics curricula were more or less static until October 5, 1957, when, in the middle of the cold war, Russia launched Sputnik. The fact that Russia had taken an early lead in the so-called "space race" led to massive attempts to catch up on the part of the American mathematics and science enterprise. In science, this gave us what are known as the "alphabet curricula" – with post Sputnik reforms including BSCS (Biological Sciences Curriculum Study), Chem Study or CHEMS (Chemical Education Material Study), PSSC (Physical Science Study Committee), ISCS (Intermediate Science Curriculum Study), IPS (Introductory Physical Science), Harvard Project Physics (no acronym!), ESCP (Earth Science Curriculum Project), and the New Math, which included SMSG (School Mathematics Study Group) and UICSM (University of Illinois Chicago School Mathematics program).

Although there is reason to dispute the broad brush view, The New Math has come to be seen as a failure. Teachers and parents were unfamiliar with the concepts that students were supposed to learn (meaning parents could not help with their children's homework) and there was broad negative societal reaction, as well as a decline in student performance on standardized tests of basic skills. In a significant pendulum swing, US curricula moved "back to basics" in the 1970s – but the net result of focusing on isolated skills was that students never learned to engage in problem-solving and were no better at basics than at the beginning of the 1970s. In response, the National Council of Teachers of Mathematics (1980) issued *An Agenda for Action*, whose first recommendation was that "Problem solving be the focus of school mathematics in the 1980s." In this, the mathematics education community meant problem-solving in the spirit of George Pólya, whose pioneering volume *How to Solve It* (Pólya, 1945, followed over time by Pólya, 1954, 1962, 1965/1981), opened up the idea of "heuristic" problem-solving strategies. Heuristics were rules of thumb such as: If you cannot solve the given problem, establish subgoals or try to break the original problem into more workable subproblems; or, consider solving easier related problems and then employing the methods or results from the easier problems to solve the original problem. Heuristic methods did not offer guarantees, but they enhanced the likelihood of making progress on, or solving, challenging problems. The challenge was that no mechanism had been found for Pólya's ideas to make their way successfully into classroom practice.

Thanks to the advent of cognitive science in the 1970s and 1980s, new methods became available to explore the nature of mathematical thinking and problem-solving – specifically, taking a process view of problem-solving (researching what people did that enabled them to be successful at solving problems) rather than focusing largely on test scores as evidence of successful interventions. The framework discussed below was available by the mid-1980s (see Schoenfeld, 1985, 1992).

What Matters in Mathematical Thinking and Problem-Solving?

The overarching issue with regard to problem-solving is: what are the key determinants of an individual's success or failure as they attempt to solve a challenging mathematical problem? The literature indicates that there are four: (1) the resources (including mathematical content, processes, and practices) at their disposal; (2) their access to and ability to use Pólya-like problem-solving strategies; (3) their metacognitive performance, specifically monitoring and self-regulation; and (4) their mathematical belief systems, which shape the ways in which they engage with mathematics.

Mathematical Resources (Including Content, Processes, and Practices)

Mathematical content has long been the focus of curricular attention. There have been periodic reports recommending mathematics curricular content since at least the late nineteenth century. Such reports reflect contemporary conceptions of essential mathematical understandings. These evolve with the times, for example, with an increasing focus on statistics and probability in the latter part of the twentieth century. A major insight from the cognitive revolution was that mathematical *processes* are as fundamental a part of doing mathematics as mathematical *content*. This insight made its way from the realm of research into practice when the National Council of Teachers of Mathematics published the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), commonly known as "the Standards." The Standards, which catalyzed what has been called the standards movement in the USA – within a decade of their publication, standards volumes were created for all major academic disciplines – highlighted the importance of four process standards. The list of standards for every grade band in NCTM (1989) began with mathematics as problem-solving, mathematics as communication, mathematics as reasoning, and mathematical connections. NCTM made the cognitively accurate but politically controversial case (Schoenfeld, 2004) that those process standards should be the leitmotifs of instruction at all grade levels.

Over the decades since the focus on mathematical processes began to flourish in the 1970s, there have been continuous refinements of the initial understandings. NCTM's (2000) *Principles and Standards for School Mathematics* added "representation" as a key mathematical process. The Common Core State Standards (2010) emphasized "practices" – the ways people engage with mathematics, individually and collectively – over processes, the actions taken while doing mathematics. Progress continues to be made on elaborating "mathematical habits of mind" or "productive patterns of mathematical thinking" (Schoenfeld, 2017). Yet, there are many challenges – see Part 3 of this chapter.

Problem-Solving Strategies

The key point about heuristics, or problem-solving strategies as characterized by Pólya, is that they are rules of thumb for making progress on problems that one has not been taught to solve. Specifically, whether or not a mathematical task is a "problem" does not reside in the task itself, but the relationship between the individual and the task: what is a problem for one person may simply be an exercise for another. Consider a task such as "what is the sum of the first 51 odd numbers?" For a student who is unfamiliar with basic problem-solving strategies, this can be a challenge. For a student who has just been taught the strategy "when a problem has an integer parameter n, either explicitly or implicitly, build a table with the values $n = 1, 2, 3, 4, \ldots$ and look for a pattern" and who is given that task on homework or on a unit test, the task is an exercise, not a problem. This raises a paradox for instruction and assessment: if such strategies are taught and tested as *content*, then students are not being taught and tested on problem-solving.

That said, there is ample evidence that problem-solving can be taught, in at least two ways. One way is to be explicit at first, identifying and teaching heuristic strategies as content, but then to remove that scaffolding, while challenging students with more complex tasks for which many different strategies may be useful. The second is to offer a wide variety of challenges and provide some scaffolding along the way, highlighting the strategies as leitmotifs of instruction.

Both of these approaches can be implemented successfully. Schoenfeld's (1985, 1992) work took the former approach. Final examinations in his problem-solving course had three parts: problems similar to those used in the course, problems that could be solved by the same methods but that did not at all resemble those used in instruction, and problems unlike those in the course, taken from books documenting problem competitions. Students consistently did well on all three parts of the exam. The problem-solving lessons from the Mathematics Assessment Project (https://www.map.mathshell.org/) took the latter approach, offering collections of rich problems that collectively scaffolded students' skills at grappling with tasks they had not been shown how to solve. There too, student performance improved significantly.

The issue, then, is how to get such instruction into the mainstream – a function of curriculum, assessment, and professional development discussed in Part 3.

Metacognition: Monitoring and Self-Regulation

It is essential to remember that problem-solving is a process, something that unfolds with time. A central question, then, is how that time – a limited resource – is used. Figure 2, a timeline representation of two students' work on a challenging task, exemplifies the issue. The students read the problem, selected a direction to pursue, and spent the balance of their allotted time working in that direction. As it happens, the direction the students chose to work in was not productive. The fact that they did not reconsider their approach *guaranteed* that they would fail to solve the problem. This exemplifies the issue of monitoring and self-regulation, a part of metacognition (Brown, 1978; Schoenfeld, 1987). Effective problem solvers keep track of how their efforts are going and adjust accordingly. They truncate fruitless attempts, look for alternatives, revisit assumptions, and more. Doing so does not guarantee that one will find a solution, but at least it keeps the door open to finding one.

Figure 3 shows the timeline graph of two students working a problem after having finished a course in problem-solving. Each inverted black triangle indicates a moment of monitoring and self-regulation. When they started working on the problem, the students jumped into a solution attempt. They paused after 2 min, ultimately deciding that they needed to be more systematic. They made a plan and started to implement it. They expressed some reservations about their progress when they were 6 and 7 min into their work and, at 11 min in, decided to rethink their approach. They then found a new approach and solved the problem. The key point here is not simply that they succeeded. It is that, absent monitoring and self-reflection, the students' timeline graph might well have resembled Fig. 2. By virtue of monitoring and self-regulation, they gave themselves the opportunity to use other knowledge.

Such metacognitive skills can be learned when instruction focuses on them. That was the case for the students whose solution generated Fig. 3. There was a poster on the classroom wall with the questions in Fig. 4, and students were frequently asked to

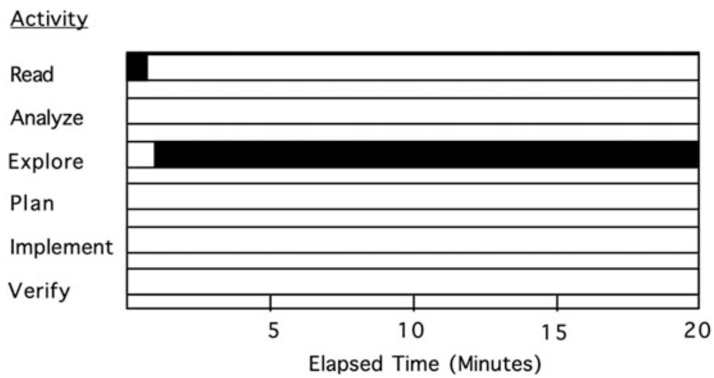

Fig. 2 A timeline graph of a typical student attempt to solve a nonstandard problem. (Reproduced with permission from Schoenfeld (1985))

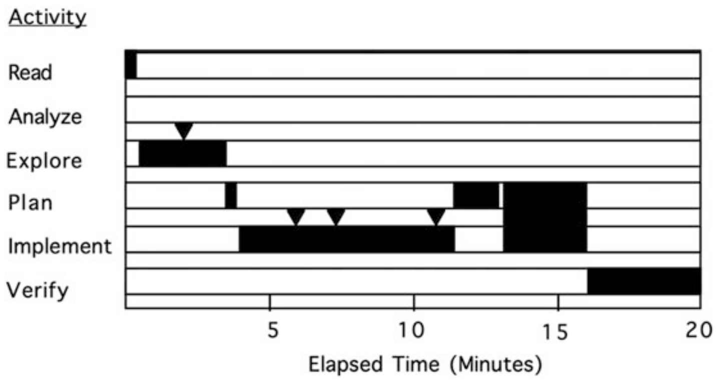

Fig. 3 A timeline graph of two students working a problem after having finished a course in problem-solving. (Reproduced with permission from Schoenfeld (1985))

What (exactly) are you doing? (Can you describe it precisely?) Why are you doing it? (How does it fit into the solution?) How does it help you? (What will you do with the outcome when you obtain it?)

Fig. 4 Metacognitive questions. (Reproduced with permission from Schoenfeld 1992)

address them. Ultimately, they internalized those questions. When that happened, it shaped their problem-solving.

The question in general is, when in typical students' experience are such understandings taught and learned? The answer, in general: it is not.

Belief Systems

What we take to be true shapes what we perceive and how we act. And, what each of us takes to be true (our belief systems) is shaped by our personal history and experience. That is the case whether those belief systems are religious, political, or about other aspects of one's experience such as the nature of mathematics or physics and how one goes about engaging in them (see, e.g., DiSessa, 1993, Schoenfeld, 1985). In physics, the constituent elements of belief systems (referred to by DiSessa, 1993 as p-prims or phenomenological primitives) are shaped largely by experience. When you stop pushing a lawnmower or other heavy object, the object stops moving. Do that enough times, and you abstract the p-prim that the continuous exertion of force is required to keep an object in motion. Similarly, one might come to believe that objects move in the direction of the forces applied to them (true of a soccer ball kicked from scratch, but not necessarily true of a soccer ball already in motion). Such abstractions from experience may be context specific and partially true; they may evolve with further experience and teaching. They combine into "personal epistemologies" (Elby & Hammer, 2010) that shape what one perceives and how one interprets the physical surround.

In the case of physics, initial understandings are grounded in one's ongoing experiences in the physical world. In the case of mathematics, school experience is the major factor that shapes what people come to believe. As Lampert notes,

> Commonly, mathematics is associated with certainty; knowing it, with being able to get the right answer, quickly... These cultural assumptions are shaped by school experience, in which *doing* mathematics means following the rules laid down by the teacher; *knowing* mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical *truth is determined* when the answer is ratified by the teacher. Beliefs about how to do mathematics and what it means to know it in school are acquired through years of watching, listening, and practicing. (Lampert, 1990, p. 31)

Such beliefs, and others (see the discussions in Parts 2 and 3), are consequential. Among the problematic student beliefs about mathematics that have been well documented, one finds the following: Mathematics problems have one and only one right answer; there is only one proper way to solve any mathematics problem (usually the rule the teacher has most recently demonstrated to the class); ordinary students cannot expect to understand mathematics, they expect simply to memorize it and apply what they have learned mechanically and without understanding; mathematics is a solitary activity done by individuals in isolation; students who have understood the mathematics they have studied will be able to solve any assigned problem in 5 min or less; the mathematics learned in school has little or nothing to do with the real world; formal proof is irrelevant to processes of discovery or invention; and, particular ethnic or racial groups are inherently good (or bad) at mathematics. These beliefs are important because students who hold them will act accordingly. The students who believe all problems can be solved in 5 min or less will stop working on problems after a few minutes, even if continuing to work on them might yield solutions. The students who believe that school mathematics does not apply to real-world phenomena will fail to use mathematical knowledge when it could be applied usefully. The students who do not expect mathematics to make sense will write down unreasonable answers to questions without stopping to question them. And so on. Moreover, to foreshadow Parts 2 and 3: members of ethnic or racial groups that are stereotypically considered poor at mathematics may be positioned as such. They may well internalize such beliefs, with comparably powerful consequences.

There is an alternative. If mathematics is approached as a sensemaking discipline, in which students engage in activities in which they themselves develop mathematical connections and insights, they can come to see mathematics as something they can understand and make their own. This, however, requires a different kind of mathematics classroom than is common at present. In such classrooms, students are not taught simply by the method of "demonstrate and practice," which places authority outside the students. In contrast, students are presented with challenges that are within reach and given opportunities to conjecture and reason their ways to defensible conclusions.

As one example, Mason et al. (1982) define the sequence of argumentation that leads to a compelling argument as "Convince yourself. Convince a friend. Convince a skeptic." They note that there is a huge change in the student stance toward mathematics when, instead of being presented a task in the form "Prove that X is true" the student is asked, "My friend says X is true. Is she right? If so, explain why. If not, give an example to demonstrate that she isn't." The latter formulation places the burden of sense making and argumentation on the students' shoulders. Aspects of inquiry-based learning and mathematical modeling are aimed at similar mathematical goals. So are the task- and discourse-centered Formative Assessment Lessons (FALs) produced by the Mathematics Assessment Project (2020). Consider, for example, the conjecturing, discussing, and argumentation involved when students engage collaboratively with the tasks in Fig. 5, taken from the FAL named, "Evaluating statements about length and area." Evidence from the literature, discussed below, indicates that ongoing engagement with such tasks – that is, ongoing opportunities to explore, conjecture, and justify one's thinking about mathematical objects – has significant positive impact. The challenge is to provide such opportunities at scale.

To sum up this part of the chapter, students' success in mathematical thinking and problem-solving depends on four things: the mathematical resources at their disposal, including content knowledge, mathematical processes and practices; the degree to which they have come to grips with problem-solving strategies; the degree to which they have developed effective metacognitive tools such as monitoring and self-regulation; and the beliefs about themselves and mathematics that they have developed. The key questions, then, are what opportunities they have to develop such understandings in productive ways. These opportunities – more precisely, the consistent lack of such opportunities and the reasons for them – are discussed in Part 3.

Part 2. The Learning Environment

Part 1 focused on the kinds of mathematical understandings one hopes students will develop. Here we expand the scope of discussion to consider the impact of mathematics learning environments, above and beyond the mathematics students learn. The reason is that mathematics teaching has an impact on much more than students' mathematics knowledge. One needs only to mention the phrase "math anxiety" – which, as noted, registered "about 93,800,000 results" on Google – to make the point. (By way of comparison, that is approximately one fourth of the "about 359,000,000 results" Google provides for the topic "mathematics teaching and

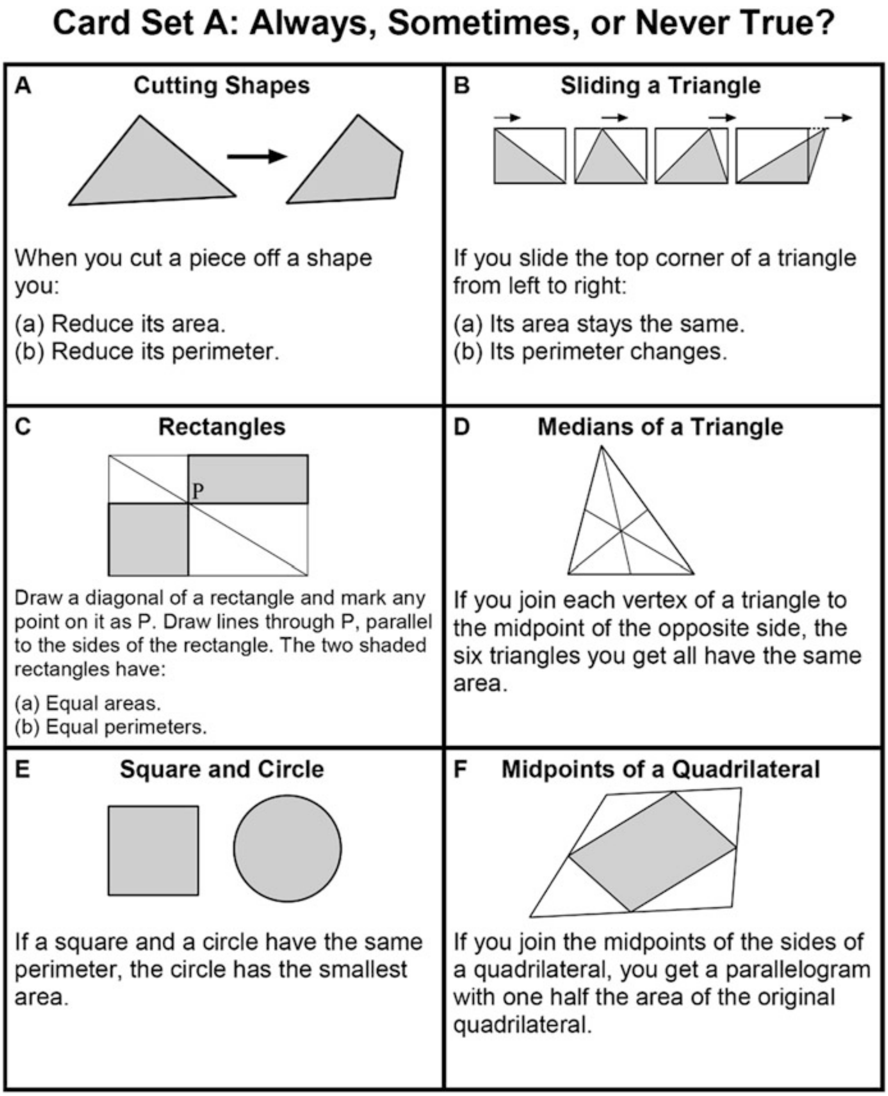

Fig. 5 Group discussion questions from the Formative Assessment Lesson “Evaluating statements about length and area.” Reprinted with permission

learning.”) More broadly, mathematics classrooms are the primary locales in which people’s mathematics identities are shaped.

Note the shift in framing from a focus on teaching to the nature of learning environments. That is because the key question from the student’s point of view is not “what does the teacher do?” but “how do I experience mathematics in the classroom?” This shift in focus from teacher actions to student experiences in no way minimizes the importance of what the teacher does, but it broadens its scope –

the issue being, what are the properties of the learning environment that the teacher has shaped, and that the student experiences?

What Is "Ambitious Instruction" or "Teaching for Robust Understanding"?

We proceed to a general framework in two stages. The first is to highlight the central goals or priorities for "powerful" or "ambitious" (and to be specific, equitable) mathematics instruction. The second is to ask, what are the attributes of mathematics learning environments that support those goals?

That students should come to be powerful mathematical thinkers and problem solvers is a given. But which students? "All students" is an immediate answer. However, in the face of a wide range of data spanning at least half a century, saying "all students" without elaboration can be seen as rhetorical in the same sense that "all lives matter" is now seen as a form of denial and negation in response to the statement that "Black lives matter." Over the past 50 years, the questions of who is or is not well served by mathematics, what the consequences are, and how to address such disparities, have come into increasingly sharp focus. A major part of that focus is a reconceptualization of the role of race as a construct. From the 1970s, a major focus on in data gathering has been on racial and ethnic "performance gaps" – one can see historical trends at https://cepa.stanford.edu/educational-opportunity-monitoring-project/achievement-gaps/race/. Scores disaggregated by racial and ethnic group have been a feature of analyses for a half century at least. The National Assessment of Educational Progress issues updates and trends when new assessments are produced; see, e.g., the 2019 NCES report at https://nces.ed.gov/programs/raceindicators/indicator_RCB.asp. The difficulty with this approach is that it reifies race, making it the object of statistical analyses alongside other, comparably unquestioned constructs. It allows for statements such as the following, in a highly cited paper: "The estimated model captures key patterns in the data, such as the widening of minority-[W]hite test score gaps with age and differences in the gap pattern between Hispanics and [B]lacks. We find that differences in mother's ability (as measured by AFQT) accounts for about half of the test score gap. However, home inputs also account for a significant proportion." (Todd & Wolpin, 2006, p. 91.). Such quantitative reification can be pernicious, feeding into deficit models and worse.

Within mathematics education, this kind of approach has been increasingly problematized, e.g., by scholars such as Rochelle Gutiérrez (2008), whose paper problematizing a focus on performance gaps is titled "A gap-gazing fetish in mathematics education? Problematizing research on the achievement gap." Davis and Martin (2008, p. 10) argue that "instructional practices can simultaneously reflect well intentioned motivations and contribute to the oppression of their African American students." The authors further argue that the focus of assessing African American children via comparison to White children "reveals underlying institutionally based racist assumptions about the competencies of African American students." Martin (2009) contrasts mainstream mathematics education research, policy, and practice with a view of mathematics learning and participation as racialized forms of experience. He notes that there are substantial differences between the two in their conceptualizations of race, conceptualizations of learners, research, policy, and practice orientations to race, and the very aims and goals of mathematics education research, policy, and practice. Moreover, he argues that a "colorblind" approach to instruction obscures important race-related issues of the type that are discussed below.

To sum up at this point: It is essential to focus on students' experience of the learning environment and to highlight both the mathematical affordances of the environment and the ways in which equitable access to that content is addressed, giving full attention to issues of race and ethnicity. The issue, then, is to identify what else is essential to focus on – and having done so, to examine the challenges we face and the reasons for them.

The Teaching for Robust Understanding (TRU) Framework

The Teaching for Robust Understanding (TRU) framework, outlined in Fig. 6, (Schoenfeld, 2014, 2020; see also https://truframework.org/) specifies what matters in mathematics classrooms, in a way that is comprehensive and yet manageable.

The Five Dimensions of Powerful Mathematics Classrooms

The Mathematics	Cognitive Demand	Equitable Access to Content	Agency, Ownership, and Identity	Formative Assessment
The extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful mathematical thinkers. Discussions are focused and coherent, providing opportunities to learn mathematical ideas, techniques, and perspectives, make connections, and develop productive mathematical habits of mind.	*The extent to which students have opportunities to grapple with and make sense of important mathematical ideas and their use. Students learn best when they are challenged in ways that provide room and support for growth, with task difficulty ranging from moderate to demanding. The level of challenge should be conducive to what has been called "productive struggle."*	*The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematical content being addressed by the class. Classrooms in which a small number of students get most of the "air time" are not equitable, no matter how rich the content: all students need to be involved in meaningful ways.*	*The extent to which students are provided opportunities to "walk the walk and talk the talk" – to contribute to conversations about mathematical ideas, to build on others' ideas and have others build on theirs – in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners.*	*The extent to which classroom activities elicit student thinking and subsequent interactions respond to those ideas, building on productive beginnings and addressing emerging misunderstandings. Powerful instruction "meets students where they are" and gives them opportunities to deepen their understandings.*

Fig. 6 Five dimensions of powerful mathematics classrooms. (Reproduced with permission from the TRU Observation Guide, p. 1)

The central idea underlying the framework is that the five dimensions of classroom practice highlighted in Fig. 6 are fundamentally important with regard to student outcomes: students will emerge from classrooms as powerful and empowered mathematical thinkers and problem solvers to the degree that their classroom experiences are consistent with those five dimensions.

We begin by noting that the TRU Framework encompasses the fundamental concerns of ambitious instruction discussed above, with a central focus on mathematical content and practices (Dimension 1), and race and equity (Dimensions 3 and 4, as explained below). It should be immediately clear that the brief descriptions of each dimension are mere suggestions of the full content of those dimensions – there is only room for so many words in each box. For example, Part 1 of this chapter outlined the aspects of mathematical thinking and problem-solving that one hopes students will come to learn. All of that (and more, see below) is intended by Dimension 1. Similarly, consider as examples the issues of race and equity discussed immediately above. An ambitious learning environment provides affordances for deep learning for students of all races, ethnicities, and demographic groups including gender, sexual orientation, and disability, responding to students in ways consonant with Martin's (2009) descriptions of learning as racialized experience, and affirming and developing student identities. Such intentions need to be unpacked from the telegraphic descriptions encapsulated in Fig. 4. That said, we outline the fundamental dimensions of the TRU Framework (see also http://truframework.org).

The mathematics. To begin with Dimension 1: The mathematical opportunities provided in the classroom are the "growth medium" for students' development of mathematical content, processes, and practices. As indicated in the first part of this chapter, there is far more to mathematical thinking and problem-solving than mere mastery of content. Will students have opportunities to engage in mathematical sense making, to see how mathematics fits together as a coherent whole? Or, will they experience it as a set of rules and procedures, to be memorized? Will they have opportunities to engage in the use of problem-solving strategies or, more generally powerful patterns of mathematical thinking? Will they have opportunities to develop metacognitive proficiency, and experiences that lead them to construct productive beliefs, both about themselves and the nature of mathematics? And, to anticipate the discussions that follow, what resources are there, for teacher and students, to make those opportunities available in ways that connect to the funds of knowledge that students bring to the classroom?

Equitable access, agency, ownership, and identity. To continue with the essentials at the beginning of this section, consider Dimensions 3 and 4, which subsume issues of equity, race, and diversity. Dimension 3 concerns equitable access to central mathematics content and practices. It may be the case that there are ample, rich mathematical opportunities in a classroom. But who gets to take advantage of them? Does the teacher consistently call on a select few students to advance the mathematical agenda? Are there "ability groups" or tracking, in which some students are engaged in challenging tasks, and others are not held to the same standard? For example, "differentiated instruction" is defined as follows:

> Differentiated classrooms have also been described as ones that respond to student variety in readiness levels, interests, and learning profiles. It is a classroom that includes and allows all students to be successful. To do this, a teacher sets different expectations for task completion for students, specifically based upon their individual needs. (https://en.wikipedia.org/wiki/Differentiated_instruction)

Such practices, with the best of intentions, may systematically sideline students and deprive them of meaningful mathematical experiences. Are there systematic inequities, perhaps unrecognized by the teacher? *How schools shortchange girls* (AAUW 1992) revealed that in many classrooms, girls are called upon less frequently and asked less difficult questions. Are there racial disparities, perhaps similarly unrecognized? How are students positioned? Is it in classic deficit-oriented ways or is it in ways that build on their strengths (Horn, 2007; Nasir et al., 2014)? Does the learning environment provide affordances for students in cultural and curricular ways, connecting to and building on the wide range of knowledge and understandings they bring with them into the classroom (Bang & Medin, 2010, Cohen et al., 1999, Ladson-Billings, 1994, Gresalfi et al., 2009; Rosebery et al. 2010)?

The issues here are more subtle than, for example, issues of turn-taking. A teacher might use a randomizing device for calling on students, which would appear to distribute opportunities equitably. But what about the curriculum itself? To make the point by analogy, there have been changes in the literary canon over the past few decades involving the inclusion of a much broader range of high quality literature than previously. The result is that a much broader range of students can see themselves and their lives reflected in the literature, and that all students engage with a much more substantial range of exceptional literature, with a broader foundation. (This, of course, is only the tip of the iceberg: for a discussion of race and reading comprehension, see Lee, 2016.) Where are the mathematics curriculum analogs? There are some (e.g., Gutstein, 2006; Gutstein & Peterson, 2005), but for the most part, the mathematics curriculum itself is exclusive rather than inclusive (D'Ambrosio et al., 2013; Martin, 2009).

Equitable and meaningful opportunities to participate are only part of the story. The issues addressed in Dimension 4 – agency, ownership, and identity – have to do with the individual's relationship to mathematics, in the classroom but also shaped by history and culture. Mathematics anxiety, often catalyzed by classroom experiences, is one example of people's relationship with mathematics. Pervasive in the West is the idea that some people are destined to be good at mathematics and that some are not – that some have the "math gene" and some don't. Similarly, racial stereotypes and narratives paint pictures of who can and can't do mathematics, and those societal stories affect both perception and performance (Martin, 2009, Nasir & Shah, 2011; Shah 2017). All of these factors shape people's mathematical senses of self and their actions, even before they enter the classroom. And, classroom discourse tends to reify them (Louie, 2019).

Given that students experience mathematics primarily through mathematics instruction, it is in the mathematics classroom that those mathematical senses of self are more fully developed and reified. What opportunities does any particular student have to get their ideas into the mix? This is, of course, a function of how students are positioned (Herbel-Eisenmann et al., 2015; McDermott, 1996); how they are invited, both tacitly and explicitly, to participate in classroom conversations; and what affordances the curriculum offers them to engage in mathematical sense making. How are students invited into mathematical dialogue? How are their contributions received? Are they simply evaluated as right or wrong? Or, are those contributions seen as potentially valuable – and examined, reflected upon, and built on? Do students have opportunities to examine, reflect, and build on the ideas of others? If the learning environment supports this kind of "accountable talk" (Institute for Research on Learning, 2011) on an ongoing basis, students have the opportunity to develop a sense of mathematical *agency* – a sense that they can be meaningful contributors to the mathematical enterprise – and a willingness to contribute to it. Producing and refining mathematical arguments, both orally and in writing, gives students a sense of *authority* in the sense of having "authored" the mathematics (Engle & Conant, 2002), and a sense of *ownership* over the content. Ultimately the question is, do people see themselves as being mathematically capable, or do they shy away from mathematics because it is simply not a part of who they are? This is the issue of *mathematical identity* (Wenger, 1998). As noted in the previous paragraph, this is not solely a function of classroom discourse: it is a function of the cultural affordances offered by curriculum and classroom culture ("big D Discourse" in the sense of Gee, 2014); thus culturally sensitive discourse and materials are essential. Students' experiences in classrooms are the primary determinants of agency, ownership, and identity, although of course the influences of the larger society pervade the classroom as well (see Part 3 below).

Cognitive demand and formative assessment. As indicated in the preceding discussion, TRU Dimensions 1, 3, and 4 represent the fundamental outcomes desired for school mathematics: that every student engages with deep mathematical content, processes, and practices in ways that support the student in developing a strong sense of mathematical agency, ownership, and identity. The challenge is, how can this be made to happen in an ongoing way? That is the supporting role of Dimensions 2 and 5, cognitive demand and formative assessment.

Meaningful learning occurs when students are stretched. If there is no challenge in the tasks they are working there is no real learning. Likewise, there is no learning if tasks are significantly out of reach. The design challenge is to find tasks and activities that provide students with meaningful opportunities for learning and that support that learning through active engagement with the content. Note that the word "meaningful" is central, and interacts strongly with the equity-related Dimensions 3 and 4: a significant part of the design issue is to find tasks that enfranchise rather than disenfranchise every student, providing opportunities for agency.

The term "cognitive demand" describes the level of difficulty, relative to what they know, of the challenges that students are asked to undertake. The idea, consistent with Vygotsky's "zone of proximal development" or ZPD (Vygotsky, 1986), is that learning takes place most effectively when students work actively in the areas that stretch them – where students can build on what they know, expanding their current understandings. To truly make sense of rich content, students need to engage in "productive struggle" (Stein and Smith 1998). One framework for thinking about different levels of challenge is Webb's (2002) Depth-of-Knowledge (DOK) framework, which might be thought of as an update of Bloom's taxonomy (Bloom et al., 1956). At the same time, it must be stressed that simple taxonomies should not be used to structure student opportunities. The whole point of "groupworthy problems" (Cohen et al., 1999) is that complex tasks with multiple points of entry support rich discourse that enables students to contribute in multiple ways and to learn from each other. A hierarchical prerequisite-based view of what students are capable of deprives them of multiple ways to engage.

A major pedagogical issue is that teachers tend to reduce cognitive demand when students find themselves frustrated and challenged, in the process depriving the students of opportunities for productive struggle and sense making (Henningsen & Stein, 1997). This can be especially pernicious when teachers make unwarranted assumptions regarding the capacity of their students, depriving them of opportunities to engage meaningfully with mathematics. (For example the focal teacher in Schoenfeld (1988), when asked if he would consider giving his class a challenging problem, responded: "Not these students. It would just confuse them. I do that with my honors students.") The instructional challenge is to provide support (e.g., heuristic advice, raising issues, suggesting things to think about) without telling students precisely what to do, situating them within their zones of proximal development and supporting them in making progress. This challenge leads us to Dimension 5, formative assessment.

Formative assessment involves orchestrating classroom conversations and activities that reveal and address the current state of students' understandings, so that the students are positioned to make use of what they know. The idea is to keep students working productively, not simply to find errors and correct them – that is, for classroom dialogue to continue to provide students with opportunities to build upon the understandings they have developed, and to address emerging misunderstandings. This process may include quizzes or tests (or well-constructed "exit tickets" or the use of student logs), but more often it includes informal information gathering, e.g., listening to what students are saying and posing questions that may, on the one hand, bring into the open incorrect assumptions or ideas that need to be challenged, and on the other hand, help the students make connections to what they do know.

In formative assessment, the information revealed about student reasoning and understanding plays a major role in shaping the classroom activities that follow (Burkhardt & Schoenfeld, 2019). In attending to student reasoning and understanding and then shaping instruction in response, teaching "becomes clearer, more focused, and more effective" (National Research Council, 2001, p. 350). Specifically, what students reveal about their thinking allows teachers to adjust the level of cognitive demand, keeping students engaged in productive struggle. When used as a means of adjusting instruction (not simply scoring students on the correctness of their work), there are consistent learning gains (Black & Wiliam, 1998; Burkhardt & Schoenfeld, 2019). An example of the impact of formative assessment (then called "diagnostic teaching" – the term formative assessment was introduced by Black and Wiliam in 1998) can be found in Birks (1987). In comparing diagnostic teaching with direct instruction, Birks graphed the pre-test, immediate post-test, and delayed post-test performance of students in both instructional conditions. The results are shown in Fig. 7. The right hand side of Fig. 7 shows the results of direct instruction. There are gains due to instruction, and then significant losses over time. In the diagnostic teaching condition (formative assessment), we see somewhat larger gains – and then very little drop-off. This substantiates the claim that meaningful learning is better retained.

Formative assessment is challenging because it calls for responding in the moment to what students do rather than following a predetermined lesson plan – and doing so without taking away too much challenge. Yet it is effective and can be scaffolded by instructional materials. The Formative Assessment Lessons (FALs) produced by the Mathematics Assessment Project (2020), for example, identify "common issues" that students find challenging and provide "suggested questions and prompts" that can help reorient students without simply telling them what to do. The FALs have been used by a wide range of teachers, with significant results. One study indicated that teaching Formative Assessment lessons for an average of

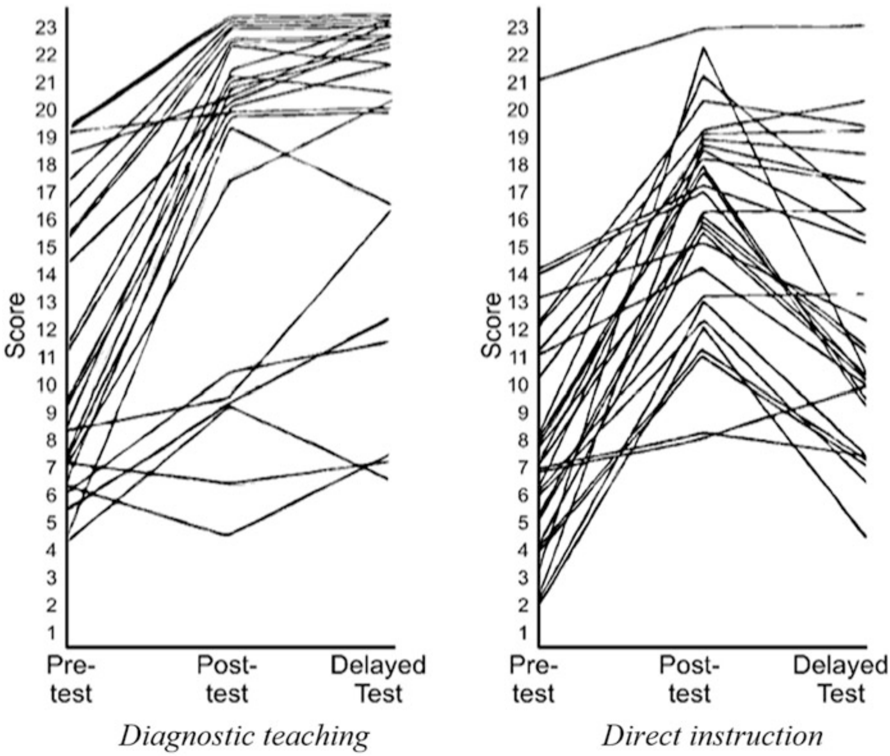

Fig. 7 Student scores on pre-, post-, and delayed post-tests

10–12 days in a year resulted in average student learning gains of 4.6 months. From using the lessons, teachers developed new pedagogical habits, specifically doing less "telling" and more probing of student understanding.

In sum, Dimensions 2 (cognitive demand) and 5 (formative assessment) of the Teaching for Robust Understanding framework provide the mechanisms for supporting the key desiderata outlined in Dimensions 1, 3, and 4: access to rich mathematics for every student, in ways that support students' agency, ownership of the mathematics, and the development of productive mathematical identities. Specifically, working with students in the moment provides opportunities to make connections and to build on the funds of knowledge that students bring to instruction. In that way, the five dimensions of TRU cohere, with each reinforcing the others.

Research underlying the framework indicates that the five dimensions of TRU are comprehensive and organized in a way that supports teachers' professional development. In theoretical terms, enough is known to make very substantial progress. Yet, far less has been made than one would like. What, then, are the obstacles?

Part 3. The Cultural Surround

Barriers to Progress

Here we address issues of curriculum, testing, and teacher support. We then consider broader issues of the cultural surround.

Issues of Curriculum and Testing

The mathematical resources deemed appropriate for students are specified in three ways. Throughout the twentieth century, the primary mechanism for curriculum specification was the textbook. Prior to 2002, the 50 states functioned independently. Each had its own department of education, which specified curricula, standards (if there were state standards), and standards-based examinations (if the state had exams); many identified textbooks that met the state's criteria. In theory, the roughly 15,000 school districts across the USA had significant latitude (subject to the constraints of college curricula and pre-college entrance exams) to teach what they wished, using whatever instructional materials they wished. The reality was otherwise, because of the ways that the textbook market functions. California, New York, and Texas, three of the most populous states in the USA, each had curricular standards, testing, and lists of approved textbooks. In those states, school districts were free to use whatever texts they wished – but the state would reimburse school districts for textbook costs only if the textbooks used were on their list of "approved" texts. For obvious fiscal reasons, most districts would only purchase books that appeared on their state's textbook adoption list. For comparably obvious reasons, major publishers made sure that their texts met the desiderata put forth by California, New York, and Texas. Textbooks were published in nationwide editions. In consequence, there was remarkable homogeneity in curricula, despite the ostensible freedom of school districts to choose whatever text materials they pleased. Given that a very large percentage of classroom interactions are driven by curricular scripts (Putnam, 1987), there was much more homogeneity than diversity for much of the twentieth century. Prior to the 1989 NCTM Standards, the focus in all major commercially available curricula was on content, with negligible attention to processes or practices; beyond content specification, there was no attention to the set of issues described in the section of this chapter titled, "What are mathematical thinking and problem solving?" That means that students – including the next generation of teachers – had little exposure to such ideas.

The NCTM Standards were published in 1989. There is no doubt that they catalyzed a move toward problem-solving, but for a number of reasons, the impetus toward mathematical substance was less direct and positive than might have been hoped. Soon after the Standards were released, the National Science Foundation, recognizing that commercial publishers would not produce Standards-based texts on their own, issued a request for proposals to develop mathematics texts consistent with the Standards. (Producing a commercial K-8 math series required a 25 Million $US investment and commercial publishers were not willing to take that large a risk in 1990.) Ultimately the NSF supported 13 such curricula, which did at least as well as the "traditional" curricula in side-by-side comparisons. Putnam's (2003, p. 161) summary of the results of the elementary curricula serves as a summary of all of the NSF-supported curricula: "The first striking thing to note about the chapters is the overall similarity in their findings. Students in these new curricula generally perform as well as other students on traditional measures of mathematical achievement, including computational skill, and generally do better on formal and informal assessments of conceptual understanding and ability to use mathematics to solve problems." Other studies confirmed that students who learned from "reform" curricula consistently outperformed students who learned from traditional curricula on tests of conceptual understanding and problem-solving, while doing as well on traditional skills-focused measures (Schoenfeld, 2002).

That simple statement is accurate, but it masks the chaos of the decade that preceded it. The Standards had as a major goal the significant reform of the mathematics curriculum. They came into being in large measure because of a perceived crisis in mathematics education: in the 1970s and 1980s, Japan was in economic ascendency and a large part of the blame was assigned to the inadequate mathematics preparation of US students. The following may be the most-cited "crisis" quote, from *A Nation at Risk* (National Commission on Excellence in Education 2003):

> Our Nation is at risk. Our once unchallenged preeminence in commerce, industry, science, and technological innovation is being overtaken by competitors throughout the world.... The educational foundations of our society are presently being eroded by a rising tide of mediocrity that threatens our very future as a Nation and a people....
>
> If an unfriendly foreign power had attempted to impose on America the mediocre educational performance that exists today, we might well have viewed it as an act of war. As it stands, we have allowed this to happen to ourselves. We have even squandered the gains in

> student achievement made in the wake of the Sputnik challenge. Moreover, we have dismantled essential support systems which helped make those gains possible. We have, in effect, been committing an act of unthinking, unilateral educational disarmament. (National Commission on Excellence in Education, 1983, p. 1)

The national security impetus was only one of many reasons for “reform.” Goals for the Standards included a literate workforce, lifelong learning, opportunity for all, and an informed electorate – much broader goals than had been previously been ascribed to mathematics. Specific changes recommended included less practice and reliance on algorithms, less emphasis on certain kinds of proofs, and other recommendations that were seen as reducing “rigor” in the service of making mathematics more accessible to a much broader population. A massive backlash ensued, and the “math wars” followed. The wars became political: Diane Ravitch, Chester Finn, and Lynne Cheney among others inveighed against what they and others called “fuzzy math” or the “new-new math” in the national media. Ultimately US Secretary of Education Richard Riley came to the 1998 joint meetings of professional mathematics societies, making a plea to tone things down. The plea went unheeded (Schoenfeld, 2004). Moreover, although the politicization of “reform” was unique to the USA, curricula and teaching in the West did not advance a great deal. It is the case that that some of the “reform” curricula were purchased by major publishing houses, and now have nationwide distribution. However, learning to teach using these new curricula requires a substantial amount of professional development. Such support had been provided for early adopters during the development of the curricular materials, but comparable support has not been made available at scale once the reform curricula entered the commercial marketplace. Thus there have not been significant curricular and professional development supports, at scale, for the kinds of teaching and learning described in this chapter. In fact, the situation is worse than that.

To repeat Putnam’s evaluation above, the Standards-based curricula did “generally do better on formal and informal assessments of conceptual understanding and ability to use mathematics to solve problems.” Students delved more deeply into content in more robust ways, and reform pedagogies did provide students more room for sense making. That represents nontrivial progress. On the other hand, consider the aspects of mathematical thinking and problem-solving discussed in Part 1 of this chapter. The problem-solving in which students engaged was for the most part devoted to richer, often contextual, problems directly related to the content of whatever unit they were studying. Building richer and more connected understandings in these ways is definitely a step forward, but it fails to address the more general Pólya-like problem-solving approaches discussed earlier in this chapter and falls far short of the kinds of sense making described in the concluding section of Part 1. Moreover, instruction as supported by texts focuses on the content and (to some variable degree) processes of doing mathematics. Issues such as metacognition and monitoring and self-regulation go unaddressed, and beliefs about the nature of mathematics are still for the most part an unaddressed byproduct of instruction. Thus, 35 years after the full spectrum of contributors to success in mathematical thinking and problem-solving was delineated (Schoenfeld, 1985), such ideas have not made their way into mainstream practice.

In addition, the contexts of instruction have in some ways become increasingly counterproductive for the development of such understandings. The first major change came as a result of the No Child Left Behind act of 2002. NCLB mandated that every state have standards and related assessments. These assessments were high stakes: states had to set achievement targets and there were very serious consequences (students being retained in grade, teachers being dismissed, and, ultimately, schools being disbanded) if test scores consistently failed to meet expectations. The result, not surprisingly, was teaching to the test. Anything not tested was unlikely to receive instructional attention; in some states such as Texas, curricula were designed to focus on items like those on sample tests. For a number of reasons including the challenges of test design (complex tasks are difficult to score reliably and test developers want to avoid lawsuits) and the costs of grading (teacher time is expensive and machine-graded multiple choice tests are "optimal" in terms of costs), tests tended to cover what was easy to grade. Instruction followed suit. This was especially problematic in districts with large percentages of African Americans (Davis & Martin, 2008).

Also not surprisingly, there were major cheating scandals. While the news media may have focused on the fact that teachers and administrators were cheating to game the system, the fact is that under such regimes, it is the students who were cheated. When a stripped-down curriculum focuses on reproducing known answers or procedures, students (including future teachers) are deprived of all of the aspects of mathematical sense making discussed in the opening sections of this chapter. Beyond that, the belief systems they develop regarding the nature of mathematics and their own mathematical abilities are hardly likely to be positive.

Fast forward a decade, from the Bush presidency and NCLB to the Obama Presidency and the Race To the Top (RTT), formally known as the American Recovery and Reinvestment Act. By the time that RTT was enacted, the math wars had calmed down, with NCTM's 1989 Standards being supplanted by NCTM (2000), *Principles and standards for school mathematics.* The intention behind RTT was to undo some of the particularly problematic aspects of NCLB – in particular the incoherence of having 50 separate sets of state standards and standards-based examinations. One main idea was that instead of individual states applying for federal funding, as had been the case under NCLB, consortia consisting of at least five states would need to apply for RTT funding, with all of the states in each consortium agreeing to "comprehensive reform plans" using the same set of standards and assessments. Such consortium-based clustering would play a significant role in reducing entropy – if all 50 states competed for funding, there would be at most 10 sets of standards and assessments nationwide, instead of 50.

There was a short deadline for producing and submitting reform plans, and states found it difficult to put them together. Facing this challenge, the National Governors Association and the Council of Chief State School Officers banded together in an effort to offer all 50 states, on a voluntary basis, a set of standards that would meet federal criteria. That effort became known as the Common Core State Standards

Initiative (2010). Ultimately 45 states adopted the Common Core State Standards for Mathematics, or CCSSM. The comprehensive reform plans required by RTT also mandated the use of assessments corresponding to the Standards. To avoid a de facto national assessment, two assessment consortia were funded: the *Partnership for Assessment of Readiness for College and Careers (PARCC) and the Smarter Balanced Assessment Consortium (SBAC). More than 40 states signed up to use these assessments at the start of RTT, although for various reasons fewer than half were using those assessments by 2017.*

Although they did address the issues of incoherence discussed above, the form of the CCSSM can be seen as causing at least five fundamental difficulties. First, while the two volumes of NCTM standards (1989, 2000) put forth sets of standards by grade band, the CCSSM itemized content standards year by year for K-8. This change enforced some homogeneity and ameliorated curricular issues when students moved from one state or district to another – there was ostensibly the same content at each grade. However, the requirement for conforming to the CCSSM on a year-by-year basis, enforced by testing, meant that almost all of the extant textbooks, including NCTM Standards-based series, were out of compliance. Text materials in conformity with the standards did not exist, causing no slight degree of scrambling. At a 2015 meeting of the Mathematics Improvement Network, nine of the ten participating school districts indicated that they were rapidly piecing together their own curricula in order to meet the demands of the common core. In most districts, such pieced-together curricula are still in use. Given that the development of coherent NCTM standards-based curricula had taken 5–10 years – and that the teams that put them together often had years if not decades of curriculum-building experience – the net result of locally constructed curricula was a decrease in the quality and coherence of materials available to teachers and students. One of the authors of the CCSSM, which appeared in 2010, told the author of this chapter that shortly after the adoption of the Common Core his granddaughter came home with a homework assignment stamped “Common Core Mathematics” – and dated 1998. Schools were scrambling.

Second, there was a fundamental problem of granularity. The more finely specified a set of curricular desiderata, the more likely it is that when school districts design their materials the goals will be to put together a curricular scope and sequence that “covers” the standards. The net result is that big ideas get lost, and there is more of a focus on content at the level of the standards that will be tested. This can lead to some conceptual streamlining – specifically, the authors of CCSSM were concerned about the historical tendency for curricula to visit and revisit fractions in grades 4, 5, and 6, and wanted to present a clear pathway for them. At the same time, one has to wonder in general about what happens to opportunities to make connections, use multiple representations, or other aspects of mathematical robustness described in this chapter. It is not that they are not implicitly in the Standards; it is that when there is a bare-bones, unexemplified set of specs, the odds of their making their way into curricula are significantly diminished.

That leads to the third issue, which is even more fundamental. Of the 93 pages in the Common Core State Standards for Mathematics, only 3 are devoted to specifying mathematical practices. In essence, the practices are invoked but invisible – in being "everywhere" (the list of eight key practices is repeated a sidebar at the beginning of each content chapter) they are nowhere. As discussed in the first part of this chapter, issues of problem-solving, metacognition, and beliefs are challenging enough to address when they are the explicit objects of attention in curriculum design. The challenges increase by orders of magnitude when the support structure for them vanishes. In sum, although enough is known to develop and support rich curricula, what most students are exposed to today – as much as when Freudenthal (1973) referred to it as such almost 50 years ago – are the "fossilized remains" of reasoning processes. Like dinosaur skeletons, they bear partial resemblance to the real thing.

The third and fourth issues, which transcend the common core, pertain to teacher readiness to take on the challenges elaborated in the first part of this chapter and to larger social issues. They merit discussions of their own.

Issues of Teacher Support

Support for teachers and teaching varies significantly from nation to nation (and there is, of course, variation within nations as well). The discussion that follows is largely centered on what takes place in the USA; there are notable exceptions in Japan, for example, where teacher professionalism is taken very seriously, or in other parts of the world where teaching is often accorded much higher respect than in the USA. Notwithstanding these differences, it should be noted that the COVID-19 epidemic, worldwide, has revealed where teachers tend to stand in the hierarchy: once the coronavirus became widespread, teachers were simply instructed to continue instruction online, even though they had had no time to prepare in any meaningful way for virtual instruction. Then, there is the question of weighing the risk of coronavirus contagion versus the potential costs of resuming live schooling to the economy. The following headline from US News (2020) makes the current US government's priorities clear: "Teachers Could Stay in Classroom if Exposed to COVID-19. New guidance from President Donald Trump's administration declares teachers to be 'critical infrastructure workers'."

But, let us take a more distanced view and provide relevant context. Type the words "those that" into google, and the fourth suggestion that comes up in the suggestion box is "those that can't do teach," with "about 581,000,000 results." Teaching expertise is generally disdained. There is no shortage of research on teaching – the fifth edition of the *Handbook of research on teaching* (Gitomer and Bell 2016) weighs in at 1539 pages and cites thousands of papers. But, much of that research is academic, in multiple senses of the word. Depending on the state in the USA, a teaching credential can typically be obtained either as an undergraduate (with a major in teaching) or in 1 year of postgraduate study (following a bachelor's degree in some content area). Typically, beginning teachers thus have acquired some content knowledge, some psychological knowledge, some pedagogical knowledge, and undertaken a brief internship during which they have had responsibility for one classroom before they are given responsibility as full-time teachers. When they are given that responsibility, the most common configuration in which they find themselves is the "egg crate model" of schooling – each teacher in a separate compartment, isolated from others – which affords little opportunity to interact with and learn from their colleagues.

Teachers emerging from pre-service preparation are thus barely prepared for the responsibilities they are about to undertake. It is thus no surprise that, normatively speaking, the first few years of teachers' careers are spent becoming skilled at classroom management (Ryan 1986). Add to that the fact that nearly 50% of new teachers leave the profession within their first 5 years (U.S. Department of Education 2020), and we are confronted with a situation where a significant majority of teachers has not experienced the richness of mathematical thinking and problem-solving themselves (much less culturally sensitive mathematics instruction), do not have powerful curricular support for rich and culturally sensitive mathematics, and may have some of their best intentions overridden by the need to do well on standardized exams, which give mathematical thinking and problem-solving short shrift.

And what of professional development? There is, again, a substantial literature on what should happen in professional development (see, e.g., Darling-Hammond et al., 2017). At some level, researchers and professional developers know what should be done; the problem is scale. The vast majority of research documenting potentially productive practices has been conducted in small scale studies, and there are neither mechanisms nor resources for providing such services to large numbers of teachers. In consequence, what most teachers do develop is the "wisdom of practice," largely by themselves, sometimes with colleagues or professional development, sometimes by virtue of membership in professional societies.

Not surprisingly, this is inadequate – and to be clear, this is not the fault of the teachers, but of the system in which they work. Consider the demands for "ambitious teaching" laid out in this chapter. The mathematical demands of the Common Core, representing both a new organization of content and demands for coherence and sense making, connections, and mathematical practices, extend far beyond the mathematical experience of most teachers. There is negligible curricular support for problem-solving and almost none for effective metacognitive behavior. To the degree that extant texts support student engagement, they may affect student beliefs about the nature of mathematics. However, mathematics learning still takes place in the context of high stakes testing, so that external pressure can undermine teachers' wishes to spend the time necessary to engage students productively in sense making activities. In short, teachers have not been provided the supports that would enable them to offer their students a rich mathematical surround.

Note that the discussion to this point focuses just on the mathematics, the emphasis of Part 1 of this chapter. As this part of the chapter indicates, it takes a great deal more than knowing mathematics to create a robust learning environment for students. Consider equity-related concerns, specifically opportunities for students to be invited into mathematical discourse in ways in which they can relate to the ideas being discussed, make the content their own, and come to see themselves as mathematical thinkers. As noted above, "Teaching for equity" is far more challenging than simply using equity sticks; building an equitable environment and making sure all students have access to core content is a substantial challenge, currently unsupported by widely accessible curricula. Approaches such as complex instruction (Cohen et al., 1999) offer a systematic approach to ambitious instruction, but curricula with adequately many "group-worthy" problems are sparsely available. Implementation of an appropriately rich professional development program typically requires significant administrative commitment and collegial support – and even with such support, it takes time and commitment to become adept at equity focused techniques. Beyond that, there is the question of curricular materials that support students' agency, ownership, and identity. There is a paucity of materials that ground mathematical ideas in students' experiences other than somewhat universal activities such as counting. Although the idea of "funds of knowledge" (Civil, 2007) is well known, developing materials that build on cultural resources has proven a particular challenge for mathematics. Groups such as TODOS (see https://www.todos-math.org/), the Equals project (https://store.lawrencehallofscience.org/Category/equals), and Rethinking Schools (https://rethinkingschools.org/) do offer mathematical resources, but such resources are few and far between – and the lockstep constraints of standards and testing make it extremely difficult to insert such materials into school curricula. As a result, teachers who want to provide culturally relevant mathematical affordances for their students find little in the way of practical help. Moreover, patterns of bias are subtle and are challenging to recognize and address (Nasir & Shah, 2011).

The preceding comments address classroom support for teachers pertaining to mathematics and equity, Dimensions 1, 3, and 4 of the TRU Framework. Then, there are issues of formative assessment (hearing what students have to say, knowing what they have to offer, and pivoting accordingly, Dimension 5) and cognitive demand (productive struggle, having students work in their zones of proximal development, Dimension 2). There are existence proofs that teachers can be supported in opening up classrooms to student thinking and building productively on it – for example, the 100 Formative Assessment Lessons produced by the Mathematics Assessment Project (2020) have, cumulatively, been downloaded almost 8 million times and, when implemented, they have had significant impact. However, it requires teaching a number of such lessons before there is any impact, and teachers who hope to use them face the same curricular and testing constraints as were discussed above for the adoption of culturally relevant curricular materials. Thus, although employing such materials and techniques can enhance students' mathematical learning, they are infrequently found in classrooms. (And, it should be noted that any technique can be implemented in equitable or inequitable ways. Even when student thinking is aired, there are questions of who gets to contribute, who gets to evaluate those contributions and in what ways, and so on).

To sum up the classroom and school-related parts of this chapter (Parts 1 and 2): when it is framed in appropriate ways, mathematics is a coherent discipline that can be learned as a form of sense making. Unfortunately, students almost never experience it that way. Curricula tend to present the distilled results of mathematical thinking, rendering it dry, unattractive, and not personally relevant. Teachers often find themselves facing new mathematical and curricular challenges. They are given little support for addressing the full range of mathematical thinking, much less the spectrum of issues necessary to create robust learning environments for all of their students. What they do develop is often scattershot – pedagogical content knowledge does emerge, but that is a long and slow process, lengthened by a paucity of support materials or collective learning communities. As discussed below, it does not have to be this way – but given the current context, it should come as no surprise that learning and teaching mathematics are as difficult as they are. And the cultural surround exacerbates the situation.

Discussion

Two major issues of 2020 cast this discussion in sharp relief. The first, the coronavirus crisis, was discussed above. Official responses to the crisis, around the world, laid bare nations' priorities regarding schooling and learning. Universally, teachers were expected to move from in-person instruction to virtual instruction while maintaining a veneer of "business as usual." Standards did not change much, if at all; as the 2020–2021 academic year approached, teachers in the USA faced essentially the same curricular expectations (and their students faced the same exams) that they faced pre-COVID. This testifies to a degree of unreality and an unwillingness to think structurally about change. Given that the "old possible" is not possible, one should be asking, How might one take advantage of the situation to rethink some of the goals of instruction? How can society serve the needs of children whose lives have just been made that much more complex? Thinking about such options did not make it into either public or official discourse. Worse, the top priorities were largely economic, and various economies rushed to reinstate in-person instruction, resulting in resurgences of COVID. This reveals in rather stark terms where instruction in general, and the specifics of mathematics instruction, stand in terms of governing priorities.

Second, the murders of George Floyd, Breonna Taylor, Trayvon Martin, Sandra Bland, Ahmaud Arbery, and numerous other Blacks at the hands of police and white supremacists laid bare for all except those who refuse to acknowledge it the structural racism that underpins American society (Urban Institute, 2020; Wilkerson, 2020). It is not that such issues were unknown; it is that the murders and the protests they engendered made them much more difficult to ignore. The reality that many minoritized people live in a world apart from White America, with different and much more devastating expectations for quality of life (including education) has been rendered day after day in high resolution.

These issues have been known and acknowledged (by some) for some time. As noted in the introduction, the Brown v. Board of Education decision by the US Supreme Court (Orfield & Eaton, 1996) may have put an end to the official policies of "separate but (ostensibly) equal" segregated schooling, but it hardly put an end to inequalities due to structural racism (Wilkerson, 2020). Books such as Jonathan Kozol's (1992) *Savage inequalities* – a best seller – made it clear that schooling was still separate and unequal for low SES and minoritized students. Data from numerous studies show that turnover rates are significantly higher (as much as 70%) for teachers in schools serving the largest concentrations of students of color. These schools are staffed by teachers who have fewer years of experience and, often, significantly less training to teach. Teacher turnover rates are 90% higher in the top quartile of schools serving students of color than in the bottom quartile for mathematics and science teachers. The deck has long been stacked against minoritized students, low income students, students with disabilities, and more generally, students who are outside the "mainstream" in any way.

That point has been doubly emphasized in times of COVID. Consider access to technology. Despite the inconvenience of transitioning to at-home virtual schooling when school districts canceled in-person classes in Spring 2020, students with resource-rich families could retreat to well-furnished homes that provided space, technology, and Wi-Fi for them to engage with their lessons. Indeed, wealthy families around the country have moved their children to private schools and/or organized COVID learning pods – in some cases hiring teachers for their children. In contrast, students and teachers in under-resourced districts have had to scramble. Despite the significant fundraising and outreach efforts that took place over the summer of 2020, 7,000 of the Oakland, CA, Unified School District's approximately 35,000 registered students lacked equipment and hotspots on the first day of school this Fall (Edsource 2020). Moreover, access to technology is just the tip of the iceberg. In areas hit hardest by COVID, issues of safety are paramount, issues of schooling secondary. A colleague who teaches in a nearby district with a very large percentage of underserved students told the author that when a student does not appear for a virtual class, her concern is not whether the student has Internet access; it is whether the student is safe.

As indicated above, structural inequalities are perhaps the most obvious and easily documentable part of the challenge of providing high quality mathematics education for every student, but they only tell part of the story. The challenge is that the inequities percolate inward in Fig. 1, from the cultural surround to the learning environment to the student. In material terms, most teachers are under-resourced and underprepared for the tasks they are asked to undertake, with the distribution of resources that do exist being extremely inequitable. Opportunities to grow are limited, and once again, inequitably distributed. But it is not only material differences that percolate inward in Fig. 1; societal biases do as well. At the overt level, teachers in the classroom are the same people they are outside the classroom – as are the students. Biases may be subtle, but they are no less real for that reason. How students are positioned (by other students as well as by the teacher) shapes student identities, as do the opportunities students have to see themselves in the materials they study. Whatever the affordances of those materials, the teacher is faced with the challenge of crafting a learning environment that does well on the dimensions highlighted by the TRU Framework – a learning environment that is mathematically rich; that not only supports every student in engaging meaningfully with core ideas but does so in ways that enfranchise every student by providing opportunities for them to get their own ideas on the table, to make the mathematics their own, and to see themselves as mathematical thinkers. Such environments must feel safe for students, so students feel they can venture ideas, while at the same time meeting high mathematical standards. And, when students reveal what they know, the teacher and environment must respond in ways that support student growth. As if that were not challenging enough, we return to the inner circle in Fig. 1 – the mathematics students should learn. It is possible for students to experience mathematics as a sensemaking activity, in which they build ideas, make connections, and come to grips with problem-solving activities; it is the case that students can develop metacognitive competencies and productive beliefs when the environment is appropriately supportive. However, the supports for attending to these aspects of mathematical thinking are few and far between. Curricula do not support them, testing often undermines them, and teachers themselves may not have had opportunities to develop such understandings themselves. It is hardly a surprise, then, that learning and teaching mathematics are as difficult as they are.

Conclusion

This chapter posed the question, "Why Are Learning and Teaching Mathematics So Difficult?" As illustrated in Fig. 1, there are three sets of reasons. The first is that only a small part of what constitutes mathematical proficiency is addressed in curricula, standards, and assessments. For the most part, instruction focuses on content; problem-solving, aspects of metacognition, and mathematically productive beliefs, practices, and habits of mind go largely unaddressed. The second is that most classrooms fall far short of providing the kinds of learning environments that nurture knowledgeable, flexible, and resourceful mathematical thinkers – the kinds of environments characterized by the Teaching for Robust Understanding Framework. For example, issues of equitable access to content are significant, both within classrooms and at the school and district levels; similarly, students are often deprived of opportunities to engage in mathematics in ways that support a sense of mathematical agency and initiative, and the development of productive mathematical identities. The third is the fact that classrooms are microcosms of society at large: societal biases permeate the classroom, both in terms of disproportionate allocations of instructional resources and in terms of the biases that students and teachers bring into the classroom. The challenges are clear, and there is much to be done.

Acknowledgments This chapter was produced with support from the US National Science Foundation Grant 1503454, "TRUmath and Lesson Study: Supporting Fundamental and Sustainable Improvement in High School Mathematics Teaching," a partnership between the Oakland Unified School District, Mills College, the SERP Institute, and the University of California at Berkeley. It has profited immensely from comments by Abraham Arcavi, Hugh Burkhardt, Diana Casanova, Gabriel Davis, Heather Fink, Vicki Hand, Nicole Louie, Dragana Martinovic, Sandra Zuñiga Ruiz, Alyssa Sayavedra, Xinyu Wei, and Anna Weltman.

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