Chapter 29
Commentary on Part III of Mathematical Challenges For All: On Problems, Problem-Solving, and Thinking Mathematically

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Eugene Wigner's (1960) essay "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" begins with part of a quote from Bertrand Russell's (1917) essay "The study of mathematics," which I provide here in slightly extended form:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of highest excellence, is to be found in mathematics as surely as in poetry. What is best in mathematics deserves not merely to be learnt as a task, but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.

It's hard to think of a better introduction to this set of chapters. Indeed, as I faced them, I had the feeling of being like a child in a candy shop – there are so many sweet confections on offer! The chapters suggest the breadth and depth of mathematics; its coherence and connections; in modeling, a bit of its unreasonable effectiveness; and the psychological pleasures of deep engagement. And yet, there is more.

This year I am once again teaching my problem-solving course. In addition to the normal variations – the course is always different because we become a mathematical community, and the people in it are different – I find that the course itself has evolved. For a number of reasons, I am less focused on problem-solving strategies per se than I once was, although heuristic strategies (and Pólya) still receive significant attention. I also cover less, because I want my students to uncover/discover more. For reasons elaborated below, I focus more on the generative nature of

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mathematics and what it means to see mathematics as a field that is rich, deeply connected, and coherent.

In what follows I revisit some of the key ideas that have shaped my problem-solving courses through the years. I then discuss the specifics of how my students and I worked through some problems this year, and why. Ultimately, problems are the raw materials for mathematical construction; if they are rich in potential, then like fine wood, metal or gems, many different things can be made of them. The question is what might be made – and how, and why.

By way of preface, I note that all of my work on problem-solving has involved an ongoing dialectic between research and development. When I have had ideas regarding ways to help students become more effective mathematical thinkers, those ideas have been tested (whether formally or informally) in my problem-solving courses. In turn, my attempts to teach problem-solving have often caused me to re-think the implementation of those ideas, or to notice hitherto unnoticed aspects of mathematical thinking and problem-solving. There has, thus, been a natural evolution of focus in my problem-solving courses – at first, as I fleshed out a framework characterizing what matters in success in problem-solving, and later as I considered the goals of mathematics instruction more broadly.

29.1 Framing a Major Point of This Chapter

A slight historical detour by way of commentary: Pólya's written work on problem-solving strategies began with *How to Solve it* (1945) and continued with the two (1954) volumes of *Mathematics and plausible reasoning* and the two (1962, 1965/1981) volumes of *Mathematical Discovery.* Before producing these volumes, all of which emphasized strategy and, in the latter volumes, pedagogy (Pólya developed *Mathematical Discovery* for sessions for teachers), Pólya produced collections of problems that, in some ways, echo the problem collections in this section of this volume. The best-known versions date all the way back to 1925: Volumes I and II of *Aufgaben und Lehrsätze aus der Analysis* (Pólya & Szegö, 1925a, b) consist of extraordinarily rich sets of problems in calculus, the theory of functions, number theory, geometry, and more. The idea behind such thematically organized collections is that any student who works through those problems will develop a deep understanding of the content. Other notable mathematicians have done the same, e.g., Halmos (1991).

This historical fact introduces a major theme of this chapter. Pólya's earlier problem collections, and Halmos's volume, had as their primary purpose the teaching of *content*. If you manage to work the problems contained in them you will have learned a substantial amount of mathematics. While it is the case that the mathematical content of *Mathematics and Plausible Reasoning* and *Mathematical Discovery* was extremely rich, Pólya's use of the problems in those volumes was different. In those volumes, the problems were organized in such a way as to highlight aspects of mathematical thinking and problem-solving. This points to the fact that collections of problems can have different purposes. They may be used to teach mathematical content, to teach heuristic strategies, to focus on mathematical thinking more broadly, to support the development of mathematical practices, and to help students develop a sense of mathematical initiative and agency as part of their mathematical identities. Hence what matters is both the richness of the problems and the uses to which they are put. In this context, it is essential to consider pedagogical issues.

What follows is a chronological narrative, with the following leitmotif: What I emphasize is a function of what is known about mathematical thinking, and what seems to be missing.

29.2 Learning to Implement Heuristic Strategies

I began researching problem-solving in the mid-1970s. Pólya's ideas about problem-solving strategies felt right to me, although I had never been explicitly taught them. They also felt right to many mathematicians; there was no question that we used the heuristic strategies Pólya described. Yet, despite a fair amount of effort, students had not been successfully taught to do so. My early research focused on making heuristic strategies implementable.

That research consisted of a mix of experimental and observational studies aimed at understanding how to implement heuristic strategies successfully. Those first studies, motivated by contemporary work in artificial intelligence, revealed that the strategies Pólya described, such as "examining special cases" and "establishing subgoals," were far more complex than they appeared to be. Specifically, each of these strategies encompassed numerous sub-strategies. For example, as described in Chapter 3 of Schoenfeld (1985), a close analysis of problem-solving attempts showed that the general description "examining special cases" applied to situations such as the following:

- The presence of a tacit or explicit integer parameter, even in a problem as simple as "what is the sum of the first 97 odd numbers?", (where "97," tacitly, is an "n") may suggest trying values of $n = 1, 2, 3, 4, \ldots$, looking for a pattern, and verifying the pattern by induction or some other means.
- It may be possible to gain insight into the nature of problems that ask about specific features of classes of algebraic functions by focusing on examples that are easy to work with. For example, if asked about the roots of polynomials in general, one might examine easily factored polynomials - or even sets of "pre-factored" polynomials such as (x), $(x)(x - 1)$, $(x)(x - 1)(x - 2)$, and so on. Doing so allows you to focus on the roots, and not get lost in algebraic manipulations.
- In computations that call for finding the limit of iterated or recursively defined terms, it may be useful to choose initial values for the terms such as 0 or 1 (if that can be done without significant loss of generality). Manipulating numbers rather than symbols may make it easier to find the underlying pattern.

- In problems that involve geometric figures, it may be useful to see if "nice" figures help – as long as one isn't seduced into believing results that aren't general. Given a problem with quadrilaterals, why not look at squares, rectangles, parallelograms, and trapezoids first? Given problems in various orientations, why not orient them conveniently?
- … and many more.

Essentially all of the heuristic strategies described in Pólya's (1945) *How to Solve It* turned out to be characterizable in this way. The top-level name of the strategy, be it "establish subgoals, "draw a diagram," or "work backwards," was accurate, and mathematicians would typically recognize a strategy when they used it. But how the mathematicians learned to use "the strategy" was something else. What had happened, most likely, is that over time they had learned many of the relevant (unnamed) substrategies, including the contextual cues that might suggest each substrategy's use – an explicit or tacit integer parameter for the first example given above, the wish to obtain roots of carefully chosen polynomials in the second example, and so on. Most of this happened without explicit labeling, as the result of repeated experiences (To paraphrase Pólya, a device used twice becomes a method). And, once one has such methods at one's disposal, the name of the strategy makes sense.

My primary goal for instruction at that time was to provide students with the experiences that would allow them to do the same – to learn the substrategies and, cumulatively, the strategy. Thus, the initial collections of problems I gave students included sets of tasks that could be worked by the same substrategy. On the first task, I might need to tell them about the strategy, or (more typically) revoice or reframe a productive move a student had made, identifying the strategy as being important. When we discussed the second task, I might ask if they'd worked a problem using a method that had helped; in debriefing a solution once they had it, I would mention not only the strategy but some of the task features that might lead them to see commonalities between the two problems (despite some obvious differences in surface features). Over the first few weeks of encountering any substrategy the students would build up their skills both in recognizing when it might be used and in using it. In subsequent weeks I would decrease the use of problems for which that particular substrategy was useful until such tasks appeared only rarely. The idea was for students to be able to identify the relevance of the substrategy when they encountered relevant problems at random, not simply when they were practicing the substrategy itself.

I won't go into detail here (See Schoenfeld, 1985), but there is clear evidence that students learned to use the substrategies, and thus the strategies themselves. Students' problem-solving performance improved significantly on three classes of problems: problems that resembled ones we'd practiced in class, problems I knew could be solved by similar methods but that did not (on the surface) resemble problems we'd worked in class, and problems chosen from collections that did not "line up" with Pólya-like methods in any obvious ways.

Another caveat before I proceed. I emphasize the introductory phrase two paragraphs above: teaching heuristic strategies was my primary goal for instruction at the time. I was looking for an existence proof, clear evidence that students could learn heuristic strategies. At the same time, I must emphasize that my course was not taught in a laboratory; it was taught in a classroom. As a result, everything I knew as a teacher and as a mathematician came into play during instruction. In particular, I chose problems that I thought led to interesting mathematics; I induced my students to explore and to generate new problems, and so on. But a primary focus was on showing that students could learn to implement problem-solving strategies.

29.3 Metacognition: Monitoring and Self-Regulation

My early problem-solving courses were largely prescriptive in nature: I identified the problem-solving methods employed by proficient problem solvers and taught students to use them. The techniques I used to uncover such moves were largely drawn from artificial intelligence research, in that I looked for systematicity in the actions of people engaged problem-solving. Beyond skill at the level of implementing heuristics, there was the question of which strategies to try, and when – issues that I called "managerial" or "executive" strategies. A one-line encapsulation of the idea is, that one should try relatively simple (but relevant) methods before spending time on more complex methods. I had built and taught an executive strategy of this kind for techniques of integration (e.g., one should look for simple substitutions before trying integration by parts or partial fractions, and try those before using complex trigonometric substitutions), and it had proven effective. So, I built a comparable executive strategy for using heuristics. The truth is that it never felt right; it was too mechanistic. Although I showed it to my students, I never emphasized it; in discussing our work in general, what I emphasized was in line with the one-line summary given above.

A grant from the National Science Foundation in the late 1970s provided me with videotape equipment. I brought students into my office space to videotape problem-solving sessions before and after my problem-solving course, to see what differences I might find. In studying the "before" tapes, it became clear that the wrong choice of direction for a solution, unreversed, could doom students to failure. This kind of event – an inappropriate choice of initial direction that was never reconsidered – happened with astonishing regularity.

Simply put, knowledge doesn't do you any good if you don't think to use it. In Schoenfeld (1987) I describe a wide range of techniques I've used in my problem-solving courses to help students become more effective at monitoring and self-regulation. But here I want to focus on a sample problem and the way my use of it evolved.

Perhaps as early as the first iteration of my problem-solving course, I have given students a slightly modified version of "a problem of construction" from Pólya's *How to Solve It:*

> You are given the triangle on the left in Fig. 29.1, below. A friend of mine claims that she can inscribe a square in the triangle – that is, that she can find a construction, using straight-edge and compass, that results in a square, all four of whose corners lie on the sides of the triangle. Is there such a construction – or might it be impossible? Do you know for certain there's an inscribed square? Do you know for certain there's a construction that will produce it?
>
> Is there anything special about the triangle you were given? That is, suppose you did find a construction. Will it work for all triangles or only some?

Pólya uses this problem to demonstrate the use of the following strategy: "If you cannot solve the given problem, try to solve first some related problem." In this case, a key goal for the desired square is that all four of its vertices lie on the triangle. Asking for less – for three vertices to lie on the triangle – might be doable; in fact, it is easy to find more than one solution (Fig. 29.2).

There are, in fact, infinitely many solutions; and one of them must pass through the 3rd side of the triangle. See Fig. 29.3.

For Pólya, this is an introductory problem, to demonstrate how the technique of using an easier related problem can help solve the original, more challenging problem. "If the student is able to guess that the locus of the fourth corner is a straight line, he has got it" (*How to Solve it*, p. 25).

For me (and, I suspect for Pólya, when he actually taught with the problem) the problem serves multiple functions. Here are a few of the mathematical ideas that arise.

Pólya's solution focuses on a particular way of finding an easier related problem, by relaxing the condition requiring that all four corners of the square lie on the triangle. That approach already narrows the solution path quite a bit. When I ask students if they can think of an easier related problem, their first response is usually to think about altering the given figure. Might it be easier to inscribe a square in an equilateral triangle? A right triangle? An isosceles triangle? Or, what about inscribing a circle in the given triangle? Occasionally, a student will suggest turning the problem inside–out – starting with a square and putting a triangle around it.

All of these suggestions raise interesting issues, among them: do you think you can solve the easier problem? If so, how might your solution lead to a solution to the original problem? And, since there are a fair number of possible approaches to take,

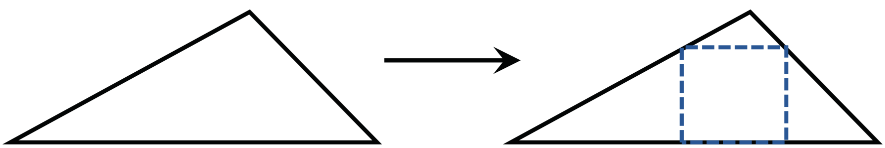


Fig. 29.1 Pólya's "problem of construction"

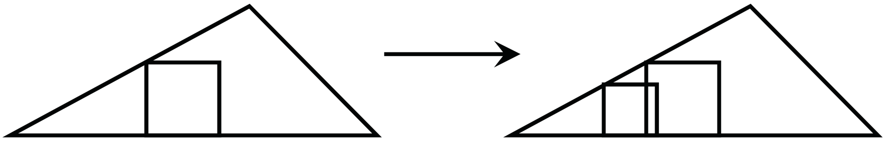


Fig. 29.2 There is more than 1 solution to the easier related problem

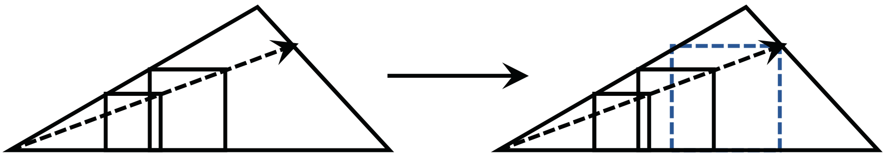


Fig. 29.3 Envisioning the solution to the original problem

how do you decide which one to try? This is our introduction to issues of metacognition.

I often let the problem "sit" until our next class meeting – has anyone made progress? Are there other ideas? If so, we pursue them. If not, I mention the generic version of Polya's suggestion: Consider the conditions of the problem and relax one of those conditions (that is, replace it with a condition that is easier to satisfy). The desired solution will be a square with four corners on the triangle. You can relax the condition of squareness to ask for a rectangle – and there are many, see Fig. 29.4 – or you can ask for fewer corners of the square to be on the triangle. Both approaches lead to existence proofs. In Fig. 29.4, one can start with short-and-wide rectangles and wind up with tall-and-thin rectangles; thus the progression must pass through a square. In Fig. 29.5, the squares with three corners on the triangle start growing inside the triangle and wind up outside it, so one of them must hit the opposite side of the triangle. In both cases, we know such a square exists. But, is there a construction that produces it? Which path should we pursue – one of these or one of the others? This makes the metacognitive challenge even more complex.

We've never managed to solve the problem for "special" triangles in a way that could be generalized; nor have we found a way to exploit the well-known method for inscribing a circle in a given triangle. And, we have yet to find a way to convert the existence proof in Fig. 29.4 into constructive proof. But, there is a lovely solution to the construction problem (first produced by my students, I hadn't known it) based on the idea of building a triangle around a square. In the tradition of classical exposition, the solution is left to the reader.

It is also worth noting that there are two very different solutions to this problem, the one in Fig. 29.5 (Pólya's solution) and the one that can be obtained by constructing a triangle similar to the original around a square. The purpose and value of problems with multiple solutions will be elaborated below.

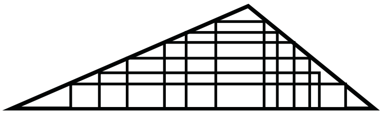


Fig. 29.4 Solutions to a different "easier related problem" lead to an existence proof

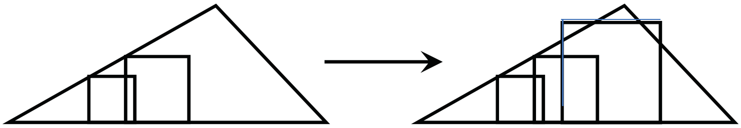


Fig. 29.5 The approach in Fig. 29.3 also leads to an existence proof

29.4 Problems as a Mechanism for Countering Unproductive Student Beliefs

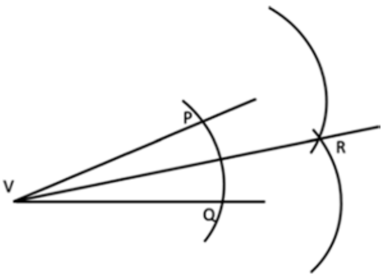
Early in my problem-solving work (See Schoenfeld, 1985) I found that many students, believing that the purpose of proofs in mathematics is to confirm in formal terms what is already understood to be true, ignored results that they had proved and made conjectures that contradicted those results. For that reason, I added a collection of (sometimes explicitly, sometimes tacitly) proof-related problems to the problem course. One year I began with a simple question: "Can anyone tell me how to bisect an angle, using a straightedge and compass?" A student quickly responded with the standard construction: (1) draw an arc from the vertex V that crosses both sides of the angle, at points P and Q; and (2) draw intersecting arcs of equal length from P and Q. Call that point of intersection R. The line from V to R bisects angle PVQ. See Fig. 29.6.

I then asked why the construction worked. There was silence at first, and then they got to work. A minute or two later, a student announced that if you drew in the line segments PR and QR, you could argue the following: PV and QV are equal because they are radii of the circle with V at the center; the compass had been kept at the same setting when creating the arcs with centers at P and Q, so PR = QR; and VR equals itself. Thus triangle PVR is congruent to triangle QVR. As corresponding parts of congruent triangles, angle PVR equals angle QVR. See Fig. 29.7.

I next asked if the students knew how to inscribe a circle in a triangle. Here too, someone remembered that the center of the desired circle lay at the intersection of the three angle bisectors. I asked why that construction worked. It took a bit longer, but before long a student produced a proof that the altitudes of the triangles drawn from the point of intersection of the three angle bisectors (the points of tangency of the inscribed circle) were all equal. At that point, a student asked, "Are you trying to tell us that proof is actually good for something?"

I said "yes, but telling you isn't enough. You have to experience it." At that point, we began to work the construction problems in Chapter 1 of Pólya's (1962/65) *Mathematical Discovery.* Once the students found, repeatedly, that deriving an

Fig. 29.6 The standard construction for bisecting an angle



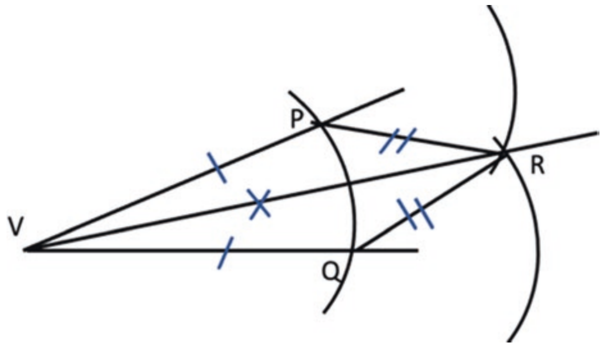


Fig. 29.7 A proof that the construction in Fig. 29.6 bisects angle V

intermediary result helped them solve a construction problem, they came to see proof as a generative tool.

The literature is replete with descriptions of counterproductive student beliefs. As Lampert wrote,

> Commonly, mathematics is associated with certainty; knowing it, with being able to get the right answer, quickly. These cultural assumptions are shaped by school experience, in which *doing* mathematics means following the rules laid down by the teacher; *knowing* mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical *truth is determined* when the answer is ratified by the teacher. Beliefs about how to do mathematics and what it means to know it in school are acquired through years of watching, listening, and practicing. (Lampert, 1990, p. 31)

As in the case of proving described above, my problem sets and my pedagogy are aimed explicitly at countering such beliefs. They do so first by providing students with enough lived experiences to provide the underpinnings of belief change. (Beliefs develop over time, as a function of experience; they must be modified in the same way.) In addition, I make my intentions and reflections explicit, because the lessons learned from experience are more likely to take hold if they are made explicit and reflected upon. Here are some sample beliefs and the problems/actions I take to address them:

- "All problems can be solved in 5 min or less." We work on problems for days and weeks.

"Mathematics is about learning to solve problems by using methods you have just been taught." We work on problems in a variety of ways – and we generate new problems. The goal is for the students to experience mathematics as a generative activity. (See the two sections that follow.)

- "Compelling patterns are enough to be convincing; the proof is just a game mathematicians play." We build up the habit of looking for patterns as a heuristic activity (à la Pólya, "let us teach guessing") … but then I throw them a curve. After a bunch of problems for which the (provable) answer is 2^n, I give them this problem:

> Suppose you pick 21 points on the boundary of a circle. You then draw all of the line segments that connect pairs of those points. If the points have been chosen so that no three of the segments intersect at the same point (that is, the circle is divided into the maximum possible number of regions), how many regions is the circle divided?

Additional examples are given in the following sections. The point is that, above and beyond "problem solving," the problems, their discussions, and the norms we cultivate in the classroom are all in the service of thinking mathematically.

29.5 Problems that Invite Multiple Solutions, an Antidote to "Answer Getting"

One of the problems I offer for discussion early in the course is this:

> Take any three-digit number and write it down twice, to make a six-digit number. (For example, the three-digit number 789 gives us the six-digit number 789,789.) I'll bet you $1.00 that the six-digit number you've just written down can be divided by 7, without leaving a remainder.
>
> OK, so I was lucky. Here's a chance to make your money back, and then some. Take the quotient that resulted from the division you just performed. I'll bet you $5.00 that quotient can be divided by 11, without leaving a remainder.
>
> OK, OK, so I was very lucky. Now you can clean up. I'll bet you $25.00 that the quotient of the division by 11 can be divided by 13, without leaving a remainder.
>
> Well, you can't win 'em all. But, you don't have to pay me if you can explain why this works.

One way to approach the problem is to note that dividing sequentially by 7, 11, and 13 is equivalent to dividing by their product – and

$$7 \times 11 \times 13 = 1001.$$

If you multiply the 3-digit number abc by 1001, you get *abc*, *abc*. Working backwards,

$$abc, abc / 1001 = abc.$$

An alternative route is to notice that the final quotient, after dividing by 7, 11, and 13,is the original number, 789. So,

$$789,789 / x = 789; x = 789,789 / 789 = 1001.$$

Another way is to say the number out loud:

$$\text{seven hundred eighty} - \text{nine thousand and seven hundred eighty} - \text{nine}$$
$$= 789(1000) + 789 = 789(1000 + 1).$$

A fourth is to ask what the number *abc*, *abc* actually means. When students are reminded to think about what base 10 notation stands for, they write out the expression

$$abc, abc = 100,000a + 10,000b + 1000c + 100a + 10b + c,$$

which they can factor as $1001(100a + 10b + c) = 1001(abc)$.

Each of these methods can be abstracted as a heuristic strategy. We discuss working forward, working backward, exploring representations, looking for patterns, and so on as various solutions emerge. In that sense, this problem fits my heuristic agenda. My favorite solution to the problem came from a graduating senior. An English major, she told me at the beginning of the course she told me that she had never liked or done well at math, and she was "giving it one last chance." After the class had generated the solutions discussed above, she raised her hand and said "I found a different solution." When I asked her, she went to the board and said,

> I didn't know what to do when I first looked at the problem, but I remembered a strategy we'd used on some other problems – if you don't know what to do, try some simple numbers and look for a pattern. The simplest 3-digit number is 001, and when I wrote 001,001 and did the divisions I saw that

$$001,001 = 7 \times 11 \times 13. \text{Then}$$

$$002,002 = 2 \times 7 \times 11 \times 13, \text{and}$$

$$003,003 = 3 \times 7 \times 11 \times 13, \text{so I got the pattern.}$$

She was proud of herself, even more so when I told her that I'd never seen that particular approach to the problem. She was energized and did well in the course.

Why work this problem so many ways? Because the goal is not to get an answer or solve a particular problem, but to perceive mathematical structure, and to make connections. Insights into the underlying structures might help develop deeper understanding of other mathematical situations. Moreover, one never knows which of the many approaches that unlock a particular problem may turn out to be useful in other situations (Problems we work on later in the semester use some of the multiple methods developed when solving earlier problems).

29.6 Transfer of Authority: Who Determines What's True?

In most mathematics courses the teacher is the sole arbiter of mathematical correctness. As Hugh Burkhardt summarizes it, "the students propose; the teacher disposes." At some point in the development of a mathematical career that has to change. Budding mathematicians come to internalize the standards of the discipline, learning to judge the correctness of arguments before they submit them for publication (A mathematician who submitted manuscripts in the hope that reviewers would determine their correctness wouldn't last very long!). Thus, the pedagogy of learning to think mathematically includes helping students come to understand that they themselves can, most of the time, determine whether or not their arguments are correct. That means helping students learn the sequence described by Mason et al. (1982) as "convince yourself, convince a friend, convince a skeptic." The problems do make a difference, however. Mathematically rich problems offer many pathways toward solutions and many ways to go astray.

Early in my problem-solving courses students will come to the board to present their work on a problem and look directly at me for affirmation. I deflect them, saying that it's the class's responsibility to question what they've done. After some time, this becomes a ritual: after finishing up at the board a student will turn to the class and say "OK, do you buy it?" With some training, the class becomes pretty good at determining whether an argument holds water. (I'm always there to do extra problematizing if need be.)

My favorite example, described in Schoenfeld (2012), is the concrete wheel problem:

> You are sitting in a room at ground level, facing a floor-to-ceiling window which is twenty feet square. A solid concrete wheel, 100 miles in diameter, is rolling down the street and is about to pass right in front of the window, from left to right. The center of the wheel is moving right at 100 miles per hour. What does the view look like from inside the room as the wheel passes by?

I will leave the solution to the reader – it's too good a problem to spoil. One of the things that's nice about the problem is that intuitions about what one would see under these circumstances vary. The wheel is moving really fast. Will the room darken almost instantaneously? Or, will it darken slowly? How long will it stay dark? What will the darkening look like – will it be like a curtain being pulled down, or will the darkness proceed from the upper left corner to the lower right corner, followed by lightness from the lower left to the upper right?

One year one group of students argued for a particular conjecture, while another group argued for a different conjecture. The argument got somewhat heated, with the rest of the class actively following the discussion. When one group prevailed, I moved to tie things up: "OK, shall I try to pull things together?" A student said "Don't bother. We got it." This was, I think, an important sign of the students' developing mathematical authority.

29.7 Problems that Are Generative: And a Bonus, for Developing a Sense of Mathematical Initiative

One of my first-day problems asks students to fill in a 3 × 3 magic square – a task that can be done by trial and error in five minutes or so. This "easy on-ramp" is one of the reasons I like the problem – all students experience success. I typically ask for a volunteer to present a solution. After they do, I ask if we're done. The class always says "yes." I respond, "No, we're not. We've only found one solution." We work through various approaches. Considering subgoals, for example, what number goes in the middle square? Or, what might the sum of each row, column, and diagonal, which we call the "magic number," be? We find solutions by working backwards (a method that allows you to find the magic number, which turns out to be 15), working forwards (listing all combinations that add up to 15), and exploiting symmetry. At that point, we have a fair number of different solutions. We've shown that there is no need for guesswork and that, save for symmetry, there is only one solution. At that point, I ask if we're done. Once again, the class says "yes." My answer, once again, is "No, we're not. To this point, you've only solved the problem I gave you to solve. If that's all mathematicians did, the field would never progress. The question now is, can we do something new and interesting grounded in what we've done? What kinds of questions can we ask?

In years gone by I've seeded the conversation by asking, "what about a magic square with the numbers 2 through 10? Or 2, 4, 6, 8, …, 18?" The class has noted that adding any constant to each cell of the 3 × 3 magic square leads to a magic square, as does multiplying each cell by a constant. So, if M is the original 3 × 3 magic square (considered as a matrix), then aM + b is also a magic square. That leads to the question, is *every* 3 × 3 magic square of the form aM+b (modulo symmetry)? When we first pondered that question, we were no longer "problem solving" or "problem posing"; we were simply doing mathematics.

Indeed, students come to recognize this as a design feature of the course. A few weeks into the course one year, I once again asked "are we done" after we had solved a problem. In mock dismay, a student threw his hands up in the air and cried "we're *never* done!" Some weeks later he asked in all seriousness why I wasn't asking "Are we done?" anymore. I answered that I didn't need to. The norms of inquiry had been established, and we were acting as a mathematical community.

29.8 This Year and the Years to Come

Through the years my problem-solving course has evolved as a function of who the students are and my perceptions of what would serve them best. As has been clear from this narrative, those perceptions evolve as my understandings of mathematical thinking and of the understandings that my students bring to the course grow and change.

Recent events have caused me, once again, to reflect on the goals of mathematics instruction. These thoughts are still in the formulation; I have been able to act on some of them, and some ideas are still prospective. The issue for me at the moment is, in what ways should people be able to use their mathematical understandings? In the context of my problem-solving courses, what are the implications for problems, problem-solving, and my pedagogy? I understand that these questions do not have unique answers, given students' varied desires and needs. I do think, however, that what follows applies to all students.

Both in my life as a private citizen and in my capacity as a member of a committee setting COVID-related safety policies for a residential program (see Schoenfeld, 2020, 2021), I have found myself wrestling with consequential "real world" problems. What policies with regard to masking, vaccination, social distancing and travel seem appropriate for our resident population? In another more personal set of issues, as someone with adult onset diabetes, I have been keeping track of my blood sugar levels for more than 20 years. Different foods affect one's blood sugar levels in different ways; the goal is to have a regimen of medicines, diet, and exercise that keeps blood sugar levels within safe bounds.

Without going into detail, I'll note that there was a practical need to take on the challenge of making COVID policy decisions. The policy climate in the US was such that government recommendations were not necessarily trustworthy or consistent (there were examples of federal policy changes within a few weeks' time when no new data had emerged to warrant a policy shift) and the residential program fell into a regulatory gap. We were on our own.

One might ask what positions me to be making COVID policy (as part of a team that includes a physician)? I'm not biologically savvy, so I certainly can't be working at a level of biological mechanism. But I can build simple mathematical models. And, while policy recommendations from federal agencies may be questionable at times, those recommendations cite the papers from which the recommendations were developed. Thus I can get information for those models without knowing the details of the biology. To give you one concrete example, here is a question I posed as I was trying to understand the aerosol dispersion of COVID particles.

> Recently, I found my nose irritated by the cigarette smoke produced by a smoker who was across the street. If that aerosol irritant could bother me at a distance of 30 feet, why is 6 feet of physical distancing considered safe for COVID?

You can find my resolution of the dilemma in Schoenfeld (2021). The resolution depends on determining two things: particle size and the density of particles expelled into the air. Once you have this information for cigarette smoke and COVID-infected molecules, the rest follows. Similarly, the same considerations explain the efficacy of masking. Vaccination data are compelling; you don't need to know the underlying science to draw conclusions about their efficacy.

Similarly, dietitians' recommendations with regard to food intake tend to be categorical ("avoid or limit rice and pasta intake") while one's reactions to white or brown rice, or different pasta dishes, can differ substantially. More consequentially, different medications for diabetes treatments have limited dosage options (10 or

25 mg in one case, multiples of 10 mg in another) and there are no impact data (that is, what dosages are likely to produce what reductions in blood sugar levels). So, when doctors and patients begin a new dietary regimen, they have to proceed empirically. At a coarse-grained level, doctors rely on a test called Hb1Ac (glycated hemoglobin) which, in rough terms, indicates a person's average blood sugar level over the past month – it's the average levels that turn out to be problematic. HbA1c results are supplemented by daily blood sugar readings, which indicate immediate sugar levels and point to possible problems. When I was diagnosed with diabetes I started simple logs of my daily sugar levels. I quickly learned to distinguish the impacts of different rice and pasta dishes (more detail than dietitians' generic information could provide), and following the data allowed me to select foods that enabled me to eat happily and well. I won a bet with my doctor, with the data showing that moderate wine consumption was good for my blood sugar levels. She won a return challenge, which showed this reluctant exerciser that a daily walk improved his blood sugar levels. I now enjoy my daily walk up the Berkeley hills, my wine with dinner, and the knowledge that they're both good for me. And, my doctor and I navigated the change of medicines smoothly, learning in the process that the impact of one medicine was not proportional to dosage (see Schoenfeld, 2021 for more detail).

What positioned me to do these things? Not my scientific knowledge; I relied on my ability to search the web and triangulate results to find the information I needed. Not my mathematical knowledge, beyond the basics that any high school graduate would have. What made the difference was my sense of initiative – the sense of personal agency that enabled me to say "let me see if I can build a simple mathematical model of the situation." Where the vast majority of people would defer to external expertise (if it existed) and its limitations, I felt personally empowered enough to look into these issues on my own, using only the web and some simple mathematical tools.

Why is it that the vast majority of people would not step outside the bounds of their school knowledge to address such consequential matters? I would argue that this happens because of a particular form of learned helplessness – one that is learned in school. In traditional schooling, students are taught content and methods that are organized to solve classes of problems. The unspoken didactical contract between teacher and students is that the tasks students will be asked to work on will closely resemble the tasks students have been taught to deal with and that when the students perform adequately on those tasks, they will be declared proficient. This is, for all intents and purposes, an inward-looking and essentially closed system. The lesson students learn from it is that their knowledge is limited to what they have been taught.

My problem-solving courses have always tried to address this issue, at least in part. The very idea of heuristic strategies is that they help one approach problems that one has not been taught to solve. More broadly, as I have outlined in this chapter, my goal has consistently been to provide my students with opportunities to *do* mathematics and to develop productive mathematical habits of mind. The very notion that "we're *never* done" is deeply embedded into the pedagogy of my problem-solving courses. But my concern goes beyond this, in that the goal is to have students feel that it is *natural* to be asking questions, and to be pursuing mathematical leads that seem interesting – whether or not those leads pan out in the end. Those thoughts have been in my mind, and they did make their way into this year's instruction.

The immediate context is that I have been teaching my graduate course on mathematical thinking and problem-solving this semester. The students all have solid mathematical backgrounds, although their interests vary: some intend to be mathematics education researchers, some science educators, and some specializing in policy and measurement. On average the class spends an hour or so each week working on problems. The rest of the time is spent reading, discussing the literature, and working on course projects.

I am about to describe a somewhat meandering classroom conversation that occurred a few weeks into the class.

As in previous years, the class and I worked through the problem of the 3×3 magic square. But, mindful of the need to be more exploratory, I looked for opportunities to highlight and support opportunities for branching out. At the beginning of the class following the class in which we had obtained a number of solutions to the problem, I provided a recap to lay the groundwork for our continued discussions. The recap described the highlights of the previous discussion, including the fact that the "magic number," the sum of each row, column, and diagonal, must be divisible by 3 (This was part of our derivation of the magic number for the original magic square, which is $(1/3)(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 15$). This was followed by an open invitation: "So now the question is, what do we do that's mathematically interesting? There's the question of generativity. What kinds of interesting questions can we ask at this point?" The conversation described below lasted about a half hour.

Two suggestions were offered. One was that we might explore 4×4, 5×5, or $n \times n$ magic squares. A second was "Can we use different numbers? Maybe higher numbers, or even numbers, or something?" I suggested we look at 2 through 10, after which one student said "the sum has to be divisible by 3, right?" I check that the sum from 2 through 10 is 54, which is divisible by 3. At that point, I say, "But suppose we wanted to cheat, what's the easiest way to use the digits 2 through 10?" A student responds, "Oh yeah, you can add … oh you can add a constant to the whole thing and it works." I elaborate, noting that if you add C to each cell in the original 3 x 3 magic square, the sum of each row, column, and diagonal is 15 + 3C.

A student then asks, "Is this kind of a proof that the sum of 9 consecutive numbers is divisible by 3?" I point out that that's a separate conjecture, and write this on the board:

Is the sum of 9 consecutive numbers divisible by 3?

One student says "I think the sum of every 3 consecutive integers is divisible by 3, so the sum of 9 numbers would actually be divisible by 3." I move to unpack the argument: The straightforward way to do the sum of 3 numbers is to write them as

a, $(a + 1)$,and $(a + 2)$; the sum is $3a + 3$, which has a factor of 3. What's the clever way?" The same student responds, "$a - 1, a, a + 1 = 3a.$"

I elaborate: if you "start with the number in the middle being called a, then you get the three numbers are $a - 1$, a, and $a + 1$, and when you add them together you get the 3a directly. Now that's just a little bit of tweaking that makes a difference in terms of representation that makes where you want to go a little bit more easy." A student completes the argument by noting that each triad of the 9 consecutive integers is thus divisible by 3, so the sum is. As usual (the class is accustomed to doing things algebraically), I proceed by noting that we could write the nine numbers as a through a + 8, whose sum is $9a + 36$. I continue, "Or, you get sneaky, and you don't have to figure out the sum from 1 to 8, because if you call the middle one a, the numbers go from [at this point there is some choral support from the class] $a - 4$ to $a + 4$. Those all balance out… and you get $9a$."

The student who originally conjectured divisibility by 3 notes that if the number of consecutive terms is divisible by 3, then the sum will be – each set of 3 terms is divisible by 3. I then noted, "OK, but you're moving towards another generalization. The sum of 3 consecutive numbers turned out to be divisible by 3. The sum of 9 consecutive numbers turned out to be divisible not just by 3 but by 9. Hmmm…".

A slightly jumbled exchange ensued. The student who had conjectured divisibility by 3 earlier said, "Whenever you have any consecutive sum, where it's divided like that (pointing to the symmetric distribution on the board) then everything will cancel out, so if the number of summands has a factor divisible by 3 then the whole thing will always be divisible by 3." I say, "By 3?" and the student says "By N." I start writing on the board as I say,

Yeah, say you've got 5 consecutive integers, just call the middle one x, then they balance out, there's an $x - 1$ and an $x - 2$, and an $x + 1$ and an $x + 2$… (see Fig. 29.8)

And *that* (I point to the $x - 2$ and $x + 2$) gives you $2x$; *that* (I point to the $x - 1$ and $x + 1$) gives you $2x$; *that* (I point to the x in the middle) gives you an x, so it's $5x$.

A second student says, "Does it always work for odd numbers?"

I respond, "Well, that's what we have so far. The sum of N consecutive integers, where N is an odd number, is always divisible by N. But what does that say about even numbers?"

A number of students start to respond, but they wind up stopping half-way through what they were saying when they realize there is no "middle number" from which to make the same symmetry argument they were able to make when there was an odd number of summands.

I editorialize, "Have you noticed, by the way, that we've shifted to doing mathematics? You're no longer doing the problem that I gave you but we're doing exactly what we're supposed to do, which is that interesting thoughts lead to interesting

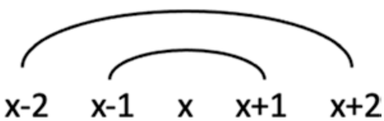


Fig. 29.8 Any five consecutive integers are "balanced" around the number in the middle

thoughts lead to… doing real mathematics. We're off in the space of conjecture, that builds off the thing that we started with."

Another student says, "I'm starting to think about even numbers. Would it be divisible by $n + 1$ if n is even?" I ask, "What do we do in the case of a conjecture like that?" and she responds, "Try a few cases."

We look at 1 + 2 = 3, which is divisible by 3. But 2 + 3 = 5, which is not. So the student's is not true. Another student says, "but it will always be odd."

After a long pause, I say, "well, a straightforward formula works for odd, but not for even… This sounds like something to leave for next week." The students laugh.

The following week I recapped what I described above and invited the students to jump back into exploring. I noted that if N was odd, the sum of N consecutive numbers was divisible by N, but that the same did not hold when n was even; in fact, the sum of two consecutive integers was always odd. One student mused that we might not be able to get any even numbers as a sum; we certainly couldn't get 2, and 4 didn't work. That was quickly put to the empirical test, and it failed:

$$1+2+3=6, \text{and} 1+2+3+4=10$$

But we were unable to obtain 8; the conjecture was modified to, "no power of 2 can be obtained as a sum of 2 or more consecutive integers." That begged to be proven but was also incomplete. We were able to get 6 = 2 × 3 and 10 = 2 × 5; what could we get? With this set of questions, the class was off and running. Someone conjectured that we could get every integer that was twice an odd number. I called a break, but the students worked right through the break. We turned to something else, but the class's e-conversation between in-person meetings was especially animated, and the class continued on its own initiative until we had a complete solution to the problem, "which integers and be expressed as the sum of two or more consecutive integers, and in how many ways can they be expressed as such a sum?"

That was a long and meandering example because our conversations were long and meandering. Let me take stock and bring things to a close.

The "consecutive sums" problem is well known. In fact, it's one of the problems I planned to assign the students later in the semester. It's fair to ask, "why to spend all that time wandering in the mathematical wilderness when you could have simply posed the problem, and perhaps even led students directly to a solution?" My response has to do with the issues of students taking mathematical initiative. There is no question that I helped to steer the conversations into mathematically productive directions, but I did so with an exceptionally light touch. Students made conjectures; they tested them; they built on what they did; they engaged in the messiness of mathematical creation, with all of the false starts that bedevil professional mathematicians when they engage in mathematics. And, they cared. They persevered because they cared, and they owned the mathematics that they produced. Moreover, they learned that they can think outside the curricular box. They learned that when issues are of interest to them, they can pursue those issues using what they know, even if they haven't been taught how to address them.

29.9 Concluding Thoughts

Let me return to the metaphor of problems as raw materials. Good problems and good problem collections are rare and wonderful things. I spent a year in the Berkeley library reading problem books before I taught the first version of my problem-solving course in the late 1970s. Of the tens of thousands of problems I examined, I found perhaps 100 that I thought merited students' attention. (By that I mean problems that are really worth working on and that students can learn valuable things from. My "problem aesthetic" is described in Schoenfeld, 2020). So, I value good problems immensely.

At the same time, problems can be used or abused, just as a piece of music can be played beautifully or badly. Wonderful raw materials can be put to good use, or they can be poorly used. What matters is how students engage in the problems. That's where pedagogy – more broadly, the creation of a learning environment in which students engage in powerful ways with mathematics – really matters.

As I hope this chapter makes clear, my thoughts on what makes for productive learning environments are very much a work in progress. In general terms, I have described the attributes of such learning environments in the Teaching for Robust Understanding (TRU) framework (see, e.g., Schoenfeld & the Teaching for Robust Understanding Project, 2016, and https://truframework.org/. In "real time," my problem-solving courses evolve as my understanding of what it means to think productively with mathematics evolves. Through the years the goals of the course have expanded to include various aspects of mathematical thinking such as heuristic strategies, metacognition, the development of productive mathematical belief systems, and powerful mathematical practices and habits of mind. They include students' development of a sense of mathematical initiative and agency and more generally powerful mathematical identities. I will, in the coming years, be grappling with questions of how to make such competencies more outward-facing, so that students will have the predilections and understandings that will enable them to use what they know more readily in contexts that matter in their personal lives.

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